SRIGAYATRI EDUCATIONAL INSTITUTIONS

INDIA

SR MPC

TRIGNOMETRY UT-2 KEY & SOLUTIONS

KEY

TRIGNOMETRIC RATIOS

1) **1**	2) **4**	3) **3**	4) **2**	5) **4**	6) **1**	7) **4**	8) **1**	9) **2**	10) **3**
11) **1**	12) **3**	13) **4**	14) **2**	15) **3**	16) **3**	17) **1**	18) **1**	19) **2**	20) **3**
21) **1**	22) **2**	23) **2**	24) **3**	25) **3**	26) **3**	27) **3**	28) **1**	29) **3**	30) **2**
31) **36**	32) **2**	33) **0.37**	34) **4**	35) **3**	36) **4**	37) **0**	38) **0.4**	39) **0**	40) **1**

COMPOUND ANGLES

1) 3	2) 1	3) 3	4) 1	5) 2	6) 3	7) 4	8) 2	9) 3	10) 1
11) 1	12) 4	13) 2	14) 1	15) 2	16) 2	17) 2	18) 3	19) 3	20) 1
21) 1	22) 2	23) 3	24) 1	25) 2	26) 1	27) 2	28) 1	29) 3	30) 1
31) 1	32) 1	33) 1	34) 1	35) 1	36) 1	37) 1	38) 1	39) 1.5	40) 1

MULTIPLE & SUBMULTIPLE ANGLES, TRANSFORMATIONS

1) **3**	2) **1**	3) **4**	4) **2**	5) **1**	6) **3**	7) **1**	8) **2**	9) **4**	10) **3**
11) **2**	12) **3**	13) **1**	14) **1**	15) **2**	16) **1**	17) **2**	18) **3**	19) **1**	20) **4**
21) **2**	22) **1**	23) **1**	24) **2**	25) **3**	26) **2**	27) **1**	28) **1**	29) **3**	30) **4**
31) **10**	32) **0**	33) **0.56**	34) **1**	35) **1.25**	36) **3**	37) **9**	38) **2.33**	39) **0.50**	40) **64**

PERIODITY, MAX & MINI VALUES

1) **2**	2) **2**	3) **4**	4) **3**	5) **2**
6) **12**	7) **-17**	8) **12**		

TRIGNOMETRIC EQUATIONS

1) **1**	2) **4**	3) **2**	4) **3**	5) **2**	6) **3**	7) **2**	8) **3**	9) **1**	10) **4**
11) **1**	12) **2**	13) **2**	14) **2**	15) **3**	16) **4**	17) **1**	18) **3**	19) **1**	20) **4**
21) **4**	22) **6**	23) **4**	24) **10**	25) **0**	26) **4.5**	27) **8**	28) **56**	29) **3**	30) **3**
31) **5**									

INVERSE TRIG. FUNCTIONS

1) **3**	2) **1**	3) **3**	4) **4**	5) **3**	6) **3**	7) **2**	8) **3**	9) **1**	10) **3**
11) **1**	12) **1**	13) **3**	14) **2**	15) **4**	16) **2**	17) **3**	18) **1**	19) **4**	20) **4**
21) **3**	22) **4**	23) **27**	24) **1.5**	25) **0**	26) **-0.5**	27) **0.6**	28) **0**	29) **-4**	30) **16**
31) **7**									

PROPERTIES OF TRIANGLES

41) **3**	42) **3**	43) **1**	44) **2**	45) **2**	46) **4**	47)	48) **3**	49) **3**	50) **4**
51) **1**	52) **3**	53) **2**	54) **1**	55) **4**	56) **4**	57) **2**	58) **3**	59) **3**	60) **2**
61) **1**	62) **4**	63) **1**	64) **2**	65) **1**	66) **3**	67) **1**	68) **3**	69) **2**	70) **1**
71) **4**	72) **3**	73) **2**	74) **3**	75) **2**	76) **3**	77) **4**	78)	79) **2**	80) **3**
81) **4**	82) **1**	83) **2**	84) **4**	85) **7**	86) **3**	87) **5**	88) **7**	89) **6.5**	90) **13**
91) **1**	92) **3**	93) **4**	94) **2**	95) **2**	96) **2**	97) **3**	98) **2**	99) **1**	100) **2**

COMPLEX NUMBERS

101) **1**	102) **2**	103) **3**	104) **4**	105) **2**	106) **2**	107) **1**	108) **3**	109) **4**	110) **3**
111) **2**	112) **2**	113) **2**	114) **4**	115) **3**	116) **4**	117) **1**	118) **4**	119) **3**	120) **3**
121) **3**	122) **2**	123) **1**	124) **3**	125) **1**	126) **4**	127) **2**	128) **3**	129) **1**	130) **1**
131) **3**	132) **4**	133) **4**	134) **2**	135) **3**	136) **3**	137) **1**	138) **2**	139) **4**	140) **2**
141) **1**	142) **2**	143) **4**	144) **4**	145) **2**	146) **3**	147) **1**	148) **1**	149) **2**	150) **3**
151) **4**	152) **2**	153) **3.5**	154) **697**	155) **-4**	156) **1**	157) **5.6**	158) **8**	159) **-1**	160) **6**
161) **4**	162) **1.5**	163) **8**	164) **-5**	165) **-1**	166) **-25**	167) **9**	168) **2**	169) **-123**	170) **3**

HINTS & SOLUTIONS

TRIGNOMETRIC ANGLES

1. $\sin\theta+\cos\theta=\frac{1}{5}$, divide with $\tan\theta$

$1+\tan\theta=\frac{1}{5}\sec\theta$

$5+5\tan\theta=\sec\theta$ S.B.S

$25+50\tan\theta+25\tan^2\theta=\sec^2\theta$

On solving $\tan\theta=-\frac{4}{3}$

If $\tan\theta=-\frac{4}{3}\Rightarrow\sin\theta+\cos\theta=\frac{1}{5}$

2. In set A $\sin x=\left(\sqrt{2}+1\right)\cos x$ or $\tan x=\sqrt{2}+1$

In set B, $\left(\sqrt{2}-1\right)\sin x=\cos x$ or $\tan x=\frac{1}{\sqrt{2}-1}\Rightarrow A=B$

3. $\sin x=1-\sin^2 x\Rightarrow\sin x=\cos^2 x\Rightarrow\cos^2 x=\sin x$

$\sin^6 x+3\sin^5 x+3\sin^4 x+\sin^3 x-2=\left(\sin^2 x+\sin x\right)^3-2=-1$

4. $\sec^4 x+\sec^2 x=10+\tan^4 x+\tan^2 x$

$\Rightarrow\sec^2 x+\tan^2 x+1=0\Rightarrow\sin^2 x=\frac{4}{5}$

5. $f_4\left(x\right)=\frac{1}{4}\left(\cos^4 x+\sin^4 x\right)\Rightarrow\frac{1}{4}-\frac{1}{2}\sin^2 x\cos^2 x$ ---- (1)

$f_6\left(x\right)=\frac{1}{6}\left(\cos^6 x+\sin^6 x\right)$ ----- (2)

$(1)-(2)=\frac{1}{12}$

6. $a\sin^2 A+b\cos^2 A=c\Rightarrow a\tan^2 A+b=c\sec^2 A$

$\Rightarrow\tan^2 A=\frac{c-b}{a-c}$ this $\tan^2 B=\frac{d-a}{b-d}$

$\Rightarrow\frac{a^2}{b^2}=\frac{\tan^2 B}{\tan^2 A}$

7. $\frac{\sin x}{|\sin x|}+\frac{\cos x}{|\tan x|}+\frac{\cot x}{|\cot x|}=f\left(x\right)$

$x\in Q_1\Rightarrow 4, x\in Q_2\Rightarrow -2, x\in Q_3\Rightarrow 0, x\in Q_4\Rightarrow -2$

$\therefore$ min value = – 2

8. $\sqrt{\sin 1}>\sin 1,\sqrt{\cos 1}>\cos 1$ by adding $\sqrt{\sin 1}+\sqrt{\cos 1}>\sin 1+\cos 1$

9. $\sum\tan A=p,\sum\tan A\tan B=0,\tan A\tan B\tan C=r$

Required value

$1+\left(\sum\tan A\right)^2-2\sum\tan A\tan B+\left(\tan A+\tan B\right)^2$

$-2\tan A\tan B\tan C\sum\tan A+\left(\tan A\tan B\tan C\right)^2$

$=1+p^2-2pr+r^2=1-\left(p-r\right)^2$

10. Let $x=\tan A-\tan B, y=\tan B-\tan C, z=\tan C-\tan A$

$x+y+z=0\Rightarrow y=-\left(x+z\right)$

S.O.B.S

$y^2 = x^2 + z^2 + 2xz$

Given $2b^2 = a^2 + c^2, b^2 = -2ac,$

$-b = \frac{2ac}{b} = \frac{2ac}{-(a+c)} = b = \frac{2ac}{a+b}$

11. $\frac{\sin^4 A}{2} + \frac{\cos^4 A}{3} = \frac{1}{5} \Rightarrow 3\sin^4 A + 2(1-\cos^2 A)^2 = \frac{6}{5}$

$\Rightarrow 25\sin^4 A - 20\sin^2 A + 4 = 0 \Rightarrow \sin^2 A = \frac{2}{5}$ and $\cos^2 A = \frac{3}{5}$

$\therefore \tan^2 A = \frac{2}{3}$

12. $\log_{\frac{1}{3}} \log 7(\sin A + a) > 0$ or $0 < \log_7(\sin A + a) < 1$ or

$1 < \sin A + a < 7 \forall x \in R$ or $1 - \sin A < a < 7 - \sin A$

It is found that 'a' should be less than the minimum of $(7 - \sin A)$ and 'a' must be greater than the minimum of $(1 - \sin A)$ $\therefore 2 < a < 6$ $\therefore$ (a = 3 or 4 or 5)

13. $\log_b^{\sin t} = x$, or $\sin t = b^x$, let $\log_b^{\cos t} = y \Rightarrow \cos t = b^y$ or $b^{2y} = \cos^2 t = 1 - \sin^2 t = 1 - b^{2x}$ or

$2y = \log_b^{1-b^{2x}}$ or $y = \frac{1}{2}\log 1 - b^{2x}$

14. $\alpha + \beta = 90° \Rightarrow \alpha = 90° - \beta \Rightarrow \sin\alpha = \cos\beta, \cos\alpha = \sin\beta$

15. $2\sin^2 135° - 1 = 2\cos^2 45 - 1 = 0$

16. $\cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B)$ here

$A = 55° + \alpha, \beta = 125° - \alpha$

17. $y = \sin^2 A + \cos^2 A + 2(\sin A \cos ecA + \cos A \sec A) + \sec^2 A + \cos ec^2 A$

$= 5 + 2 + \tan^2 A + \cot^2 A = 7 + (\tan A - \cot A)^2 + 2$

$\therefore$ minimum value = 9

18. $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$

$= 1 - 2\sin^2 x \cos^2 x \le 1$

19. $p = \sec A + \cos ecA, q = \sec A \cos ecA$ then verify

20. $\cot^4 A - 2(1 + \cot^2 A) + a^2 = 0 \Rightarrow \cot^4 A - 2\cot^2 A + a^2 - 2 = 0$

$\Rightarrow (\cot^2 A - 1)^2 = 3 - a^2 \Rightarrow 3 - a^2 \ge 0$ ($\therefore$ L.H.S ≥ 0)

$\Rightarrow a \in [-\sqrt{3}, \sqrt{3}]$ $\therefore$ integral values –1, 0, 1

21. By simplifying $\alpha^2 + \beta^2 + 2\alpha\beta = (\alpha + \beta)^2$

$\therefore$ which is independent of γ

22. $1 + \sin 2\alpha = b^2 \Rightarrow |\sin\alpha - \cos\alpha| = \sqrt{1 - \sin 2\alpha} = \sqrt{1 - (a^2 - 1)}$

23. Put $A = 45°$

24. $\alpha - \beta = \pi$

25. $\sin^2 A(\cos^2 B + \sin^2 B) + \cos^2 A(\sin^2 B + \cos^2 B)$

26. By squaring and adding $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

27. Let $\alpha+\beta=60°, \alpha-\beta=45° \Rightarrow 2\beta=15° \Rightarrow 18\beta=135^0$

28. $\sin\frac{3\pi}{2}=7, \cos\frac{3\pi}{2}=0$

29. G.P condition $b^2=ac$

$\Rightarrow (2\tan\theta+2)^2=\tan\theta(3\tan\theta+3)$

$\Rightarrow \tan^2\theta+5\tan\theta+4=0 \Rightarrow \tan\theta=-1,-4$

$\tan\theta=-4$ is only

30. $2=2^C=114$ approximately verify with values

31. $A+B=360°$ or $A+B=180° \Rightarrow \cos^2 A+\cos^2 B=1$

32. $\sin^2 x+\frac{1}{\sin^2 x}\geq 2$

33. Put $\theta=330°$ and find value

34. Let $a\sin x=b\cos x=c\tan 2x=\ell$

$\Rightarrow \sin x=\frac{\ell}{a}, \cos x=\frac{\ell}{b}, \tan 2x=\frac{\ell}{c}$

Eliminate 'ℓ', we get

$\left(a^2-b^2\right)^2=4c^2\left(a^2+b^2\right) \Rightarrow k=4$

35. $\cos\theta=\sin^2\theta \Rightarrow \cos^2\theta=\sin^4\theta$

$\left(\sin^4\theta+\sin^2\theta\right)=1$

cubing on both sides ----

36. $\sin x\left(1+\sin^2 x\right)=\cos^2 x$ squaring on both side we get

$\left(1-\cos^2 x\right)\left(2-\cos^2 x\right)^2=\cos^4 x$ then simplify

37. $\cos 70°-\sin 70°$ is negetive

38. $10=\sec^4\theta+\sec^2\theta-\tan^4\theta-\tan^2\theta$

$\Rightarrow \sec^2\theta+\tan^2\theta+1=2\sec^2\theta$

$\Rightarrow \cos^2\theta=\frac{1}{5} \Rightarrow \sin^2\theta=\frac{4}{5}$

39. Put $\theta_1=\theta_2=\theta_3=90°$

40. $\alpha+\beta=60°, \alpha-\beta=45° \Rightarrow 2\beta=15°$

COMPOUND ANGLES

1. $\sin A+\cos B=\alpha,\ \sin B+\cos A=\beta$

$\Rightarrow \sin(A+B)=\frac{\alpha^2+\beta^2-2}{2}$

2. $\alpha+\beta+\gamma=360^0, \alpha+\beta=360^0-\gamma$

Apply tan on both sides

3. $\tan(\pi\cos\theta)=\tan\left(\frac{\pi}{2}-\pi\sin\theta\right)$

$\Rightarrow \pi\cos\theta=\frac{\pi}{2}-\pi\sin\theta \Rightarrow \cos\theta+\sin\theta=\frac{1}{2}$

$\cos\left(\theta-\frac{\pi}{4}\right)=\cos\theta-\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) = \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{1}{2\sqrt{2}}$

4. $\tan\beta = \frac{2\sin\alpha\sin\gamma}{\sin(\alpha+\gamma)}$

$\Rightarrow \frac{1}{\tan\beta} = \frac{\sin(\alpha+\gamma)}{2\sin\alpha\sin\gamma}$

$\Rightarrow \frac{2}{\tan\beta} = \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\sin\alpha\sin\gamma}$

$\Rightarrow 2\cot\beta = \cot\gamma + \cot\alpha$ (A.P. series)

5. $\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}} = \left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^3$

Squaring an both sides

$\frac{1+\sin y}{1-\sin y} = \left(\frac{1+\sin x}{1-\sin x}\right)^3$

Apply componendo and dividendo

$\frac{2\sin y}{2} = \frac{(3+\sin^2 x)\sin x}{1+3\sin^2 x} \Rightarrow \frac{3+\sin^2 x}{1+3\sin^2 x} = \frac{\sin y}{\sin x}$

6. Let $\alpha = \frac{\pi - A}{4}, \beta = \frac{\pi - B}{4}, \gamma = \frac{\pi - C}{4}$

$\Rightarrow \alpha+\beta+\gamma = \frac{\pi}{2} \Rightarrow \sum \tan\alpha\tan\beta = 1$

$\Rightarrow \sum \tan^2\alpha = 1 = \sum \tan\alpha\tan\beta$

$\Rightarrow \tan\alpha = \tan\beta = \tan\gamma$

7. Apply componendo and dividendo then simplify

$\Rightarrow \sin(B-C)(\sin 2A + \sin 2B + \sin 2C) = 0$

8. $\tan P \tan C = P \Rightarrow \tan B \tan\left(\frac{3\pi}{4} - B\right) = P$

$\Rightarrow \tan^2 B + (1-P)\tan B + P = 0$

$\Delta \ge 0$ $\quad\because \tan B$ is real

9. $\sin\alpha = A[\sin\alpha\cos\beta + \cos\alpha\sin\beta]$

$\Rightarrow \sin\alpha(1 - A\cos\beta) = \cos\alpha\sin\beta$

10. $\frac{1+\tan x}{1-\tan x} = a \Rightarrow \tan x = \frac{a-1}{a+1}$ and $\sec^2 x = 1 + \tan^2 x$

11. $(1+\tan A)(1+\tan B) = 2 \Rightarrow \tan A\tan B = 1 - \tan A\tan B$

$A + B = \frac{\pi}{4} \Rightarrow \alpha + 4\alpha = \frac{\pi}{4} \Rightarrow \alpha = \frac{\pi}{4}$

12. $A+B+C = \pi \Rightarrow A+B = \pi - C$

$\Rightarrow \cot(A+B) = \cot(\pi - C) \Rightarrow \sum \cot A\cot B = 1$

13. $\cos 17^o = \cos(45^o - 28^o)$

$\Rightarrow \cos 45^o \cos 28^o + \sin 45^o \sin 28^o = \frac{k^3}{\sqrt{2}}$

14. $\cos(\alpha - \beta) = \cos[(\theta-\beta) - (\theta-\alpha)]$

$= ab + \sqrt{1-a^2}.\sqrt{1-b^2}$

$\left[\cos(\alpha-\beta) - ab\right]^2 = (1-a^2)(1-b^2)$

$\Rightarrow a^2 + b^2 = \sin^2(\alpha-\beta) + 2ab\cos(\alpha-\beta)$

15. $2\sec 2\theta = \tan\phi + \cot\phi$

$\Rightarrow \frac{2}{\cos 2\theta} = \frac{1}{\sin\phi\cos\phi} \Rightarrow 2\theta = 90 - 2\phi$ or $\theta + \phi = \frac{\pi}{4}$

16. $f(\beta) = f\left(\frac{5\pi}{4} - \alpha\right) = \frac{\cot\left(\frac{5\pi}{4} - \alpha\right)}{1 + \cot\left(\frac{5\pi}{4} - \alpha\right)} = \frac{1 + \tan\alpha}{2}$

$f(\alpha) = \frac{\cot\alpha}{1 + \cot\alpha} = \frac{1}{1 + \tan\alpha}$

$f(\alpha) - f(\beta) = \frac{1}{2}$

17. $\cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x}\right)\left(\frac{\cot x \cot z + 1}{\cot z - \cot x}\right)$

Cross multiply and simplity

$\cos 2x = \frac{\cot y + \cot z}{2}$

18. $\cos^2(\alpha+\beta) + \cos^2(\alpha-\beta) - \cos 2\alpha \cos 2\beta$

$= \cos^2(\alpha+\beta) + 1 - \sin^2(\alpha-\beta) - \cos 2\alpha\cos 2\beta$

$= 1 + \cos 2\alpha \cos 2\beta - \cos 2\alpha \cos 2\beta = 1$

19. Let $x = 3\cos\theta, y = 3\sin\theta, z = 2\cos\phi$, $t = 2\sin\phi$

$6\cos\theta\sin\phi - 6\sin\theta\cos\phi = 6 \Rightarrow \phi - \theta = 90^o$

$\phi = 90^o + \theta, x = -6\sin\theta\cos\theta = -\sin 2\theta \le 3$

20. $\tan^2 A = 2\tan^2 B + 1$ or $1 + \tan^2 A = 2(1 + \tan^2 B)$

$\Rightarrow \sec^2 A = 2\sec^2 B \Rightarrow \cos^2 B = 2\cos^2 A = 1 + \cos 2A$

$\Rightarrow \cos 2A = \cos^2 B - 1 = -\sin^2 B \Rightarrow \sin^2 B + \cos^2 A = 0$

21. Let $S = \sum_{k=1}^{100} \sin(kx)\cos(101-k)x$

$\Rightarrow s = \sin x \cos 100x + \sin 2x \cos 99x + ... + \sin 100x \cos 2x$

On writing reverse order and adding

$2s = \sin 101x + \sin 101x + ... + \sin 101x(100 times)$

$s = 50\sin(101x)$

22. $\cos\theta - \sin\theta = \frac{1}{5}$ squaring on both sides

$1 - \sin 2\theta = \frac{1}{25}$ or $\sin 2\theta = \frac{24}{25}$ or $\cos 2\theta = \frac{7}{25}$

23. $(A-B) + (B-C) + (C-A) = 0$

$\Rightarrow (A-B) + (B-C) = -(C-A)$

Apply ‘ tan ’ on both sides, cross multiply

24. $A + B + C = 180^o \Rightarrow B + C = 180 - A$

$\sin(B+C) = \sin A \Rightarrow \cos ecA.\sin A = 1$

25. $\cos(\alpha-\beta)=\cos\left(\overline{\theta-\alpha}-\overline{\theta+\beta}\right)$

26. Expand $\tan(A-B)$ and cross multiply

27. $2\cos\theta = x+\frac{1}{x}$ squaring on both sides and simplity

28. Divide with 25 and put $\cos\alpha=\frac{7}{25},\sin\alpha=\frac{24}{25}$

29. $\frac{\frac{x}{y}\tan A+\tan B}{\frac{x}{y}+1}=\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\frac{A+B}{2}$

30. $\sin^2 x+\sin x\cos x = k$

$1-\cos 2x+\sin 2x = 2k \Rightarrow \sin 2x-\cos 2x = 2k-1$

$\sin\left(2x-\frac{\pi}{4}\right)=\frac{2k-1}{\sqrt{2}}$ and $-1\le\frac{2k-1}{\sqrt{2}}\le 1$

31. $\sin\theta_1\sin\theta_2-\cos\theta_1\cos\theta_2=-1$

$\cos(\theta_1+\theta_2)=1\Rightarrow\theta_1+\theta_2=2n\pi, n\in I$

or $\frac{\theta_1}{2}+\frac{\theta_2}{2}=n\pi$

$\Rightarrow \tan\frac{\theta_1}{2}\cot\frac{\theta_2}{2}=\tan\frac{\theta_1}{2}\cot\left(n\pi-\frac{\theta_1}{2}\right)$

$=-\tan\frac{\theta_1}{2}\cot\frac{\theta_1}{2}=-1$

32. $3\sin A\cos B=\sin B\cos A\Rightarrow\cos A\sin B=\frac{3}{4}$

$\Rightarrow\sin(A+B)=1\Rightarrow C=\frac{\pi}{2}\Rightarrow B=\frac{\pi}{2}-A$

$3\tan A=\tan\left(\frac{\pi}{2}-A\right)\Rightarrow\cot^2 A=3$

33. $\tan\left(\frac{\pi}{4}-\theta\right)\tan\frac{\pi}{4}.\tan\left(\frac{\pi}{4}+\theta\right)=1$

34. $\tan\left(\frac{\pi}{2}-\frac{7\pi}{16}\right)+2\tan\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)-\cot\left(\pi-\frac{15\pi}{16}\right)$

$=\tan\frac{\pi}{16}-\cot\frac{\pi}{16}+2\tan\frac{\pi}{8}$

$=-\frac{2\cos\frac{\pi}{8}}{\sin\frac{\pi}{8}}+\frac{2\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}}=\frac{-4\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}=-4$

35. $\cos(\theta-\phi)=\sin\beta\sin\gamma$

$\cos\theta\cos\phi+\sin\theta\sin\phi=\sin\beta\sin\gamma$

$\Rightarrow\sin^2\theta\sin^2\phi=(\cos\theta\cos\phi-\sin\beta\sin\gamma)^2$

$\sin^2\alpha=\frac{\sin^2\beta+\sin^2\gamma-2\sin^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$

$\cos^2\alpha=\frac{1-\sin^2\beta-\sin^2\gamma+\sin^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$

$$\Rightarrow \tan^2\alpha = \frac{\sin^2\beta\cos^2\gamma + \cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$$

$$\Rightarrow \tan^2\beta + \tan^2\gamma \Rightarrow \tan^2\alpha - \tan^2\beta - \tan^2\gamma = 0\forall \in R$$

36. Let $x = \tan 10^o, \tan 30^o = \frac{1}{\sqrt{3}} \Rightarrow \frac{3x - x^3}{1 - 3x^2} = \frac{1}{\sqrt{3}}$

Squaring on both sides

$3t^6 - 27t^4 + 33t^2 = 1$

37. By applying componendo and dividendo

$$\frac{\sin 2\alpha}{\sin 2(\beta-\gamma)} = \frac{\sin(\gamma+\beta)}{\sin(\gamma-\beta)}$$

$$\Rightarrow \sin 2(\beta-\gamma)\sin(\beta+\gamma) + \sin 2\alpha \sin(\beta-\gamma) = 0$$

$$\Rightarrow \sin(\beta-\gamma)\left[2\cos(\beta-\gamma)\sin(\beta+\gamma) + \sin 2\alpha\right] = 0$$

$$\Rightarrow \sin(\beta-\gamma)\left[\sin 2\alpha + \sin 2\beta + \sin 2\gamma\right] = 0$$

38. If $A - B = \frac{\pi}{4}$ then $(1+\tan A)(1-\tan B) = 2$

39. $\frac{\sin(2A+B)}{\sin B} = \frac{1}{5}$

Apply componendo and dividendo

40. $\cot A\cot B + \cot B\cot C + \cot C\cot A$

$$= \frac{\sin A\sin B\sin C + \sin(A+B+C)}{\sin A\sin B\sin C}$$

MULTIPLE & SUBMULTIPLE ANGLES, TRANSFORMATION

1. Use formulae of

$$\sin A\sin(60-A)\sin(60+A) = \frac{1}{4}\sin 3A$$

$$\tan A\tan(60-A)\tan(60+A) = \tan 3A$$

2. $\frac{2\cos^2\theta - 1}{\cos^4\theta} + \frac{1}{\cos^2\theta} = 0$

$\cos^2\theta = \frac{1}{3} \Rightarrow \sin^2\theta = \frac{2}{3}$

3. $2\sec 4\alpha = 2\cos ec 4\beta$

$\sec 4\alpha = \sec\left(\frac{\pi}{2} - 4\beta\right)$

$\alpha + \beta = \frac{\pi}{8}$

4. Put $A = B = 30^0$

5. $\frac{\tan A}{\cot B} = \frac{1}{2} \Rightarrow \frac{\tan A}{1} = \frac{\cot B}{2} = k$

$\tan A = k, \cot B = 2k$

Now $(5 - 3\cos 2A)(5 - 3\cos 2B)$

$$=\left(5-3\left(\frac{1-k^2}{1+k^2}\right)\right)\left(5-3\left(\frac{1-\frac{1}{4k^2}}{1+\frac{1}{4k^2}}\right)\right)$$

$$=\frac{2\left(1+4k^2\right)}{1+k^2}\left(\frac{8k^2+8}{4k^2+1}\right)=16$$

6. $\alpha+\beta=90^0$

$\sin(\alpha+3\beta)=\sin(90+2\beta)=\cos 2\beta$

7. $-\sin 18.\sin 54=-\frac{1}{4}$

8. Use componendo dividendo method.

9. $1+\sin\frac{\pi}{n}=\frac{n}{4}$

$\sin\frac{\pi}{n}=\frac{n}{4}-1$

$\Rightarrow -1<\frac{n}{4}-1<1$

$\Rightarrow 4<n<8\Rightarrow n=5,6,7$

10. $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+2\cos\frac{\pi}{2}}}}}}}$

$=2\cos\left(\frac{\pi}{2.2^6}\right)=2\cos\left(\frac{\pi}{128}\right)$

11. $-\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}=\frac{1}{16}$

12. $\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta-\frac{\cos\theta}{|\operatorname{cosec}\theta|}-2=-1$

$\sin\theta.\cos\theta-\cos\theta|\sin\theta|=0$

$\sin\theta>0$ and $\sin\theta\neq\cos\theta$

$\therefore\ \theta\in\theta_2\Rightarrow\theta\in\left(\frac{\pi}{2},\pi\right)$

13. $1-\cos^2 A+\sin^2 B+\sin^2 C=2$

$\cos C\cos(A-B)+1-\cos^2 C=1$

$\cos C(2\cos A\cos B)=0$

$\therefore\ \angle A=90^0$ or $\angle B=90^0$ or $\angle C=90^0$

$\therefore\ \Delta ABC$ is right angled

14. If n is odd, $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n=0$

15. $\sin A=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$\frac{A}{2}=\frac{B-C}{2}\Rightarrow A+C=B$

16. $\frac{\cos x+\cos y}{\cos x-\cos y}=\frac{3}{8}$

$\tan\left(\frac{x-y}{2}\right)\tan\left(\frac{x+y}{2}\right)=\frac{3}{8}$

17. Put $x=0$ and verify the options

18. $\cos 3A+\cos 3B+\cos 3C$

$=4\left(\cos^3 A+\cos^3 B+\cos^3 C\right)-3\left(\cos A+\cos B+\cos C\right)$

$=4\left(\cos A\cos B\cos C\right)-3\left(0\right)$

$=12\cos A\cos B\cos C$

19. $k=\frac{1-2\left(\sin 10\cos 20\right)}{2\sin 10}=1$

Now $2k+3=5$

20. $16\left(\cos 2A-\cos 3A\right)$

$16\left(2\cos^2 A-1-4\cos^3 A+3\cos A\right)=11$

21. $\frac{1}{2}\left(1+\cos 20-\cos 60-\cos 40+1+\cos 100\right)$

$=\frac{3}{4}$

22. Use formula

$\cos^3 A+\cos^3\left(120-A\right)+\cos^3\left(120+A\right)=\frac{3}{4}\cos 3A$

23. Use componendo dividend method

24. $\frac{\sqrt{2}-\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right)}{\sqrt{2}\sin\left(\alpha-\frac{\pi}{4}\right)}=\tan\left(\frac{\alpha}{2}-\frac{\pi}{8}\right)$

25. $a^2+b^2=2+2\cos 2\theta=4\cos^2\theta$

Now $\frac{\cos 3\theta}{\cos\theta}=4\cos^2\theta-3=a^2+b^2-3$

26. Put $A=B=C=90°$ and verify the options

27. $2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)=-2\sqrt{3}\sin\left(\frac{B+A}{2}\right)\sin\left(\frac{B-A}{2}\right)$

$\cot\left(\frac{A-B}{2}\right)=\sqrt{3}\Rightarrow A-B=60$

Now $\sin 3A+\sin 3B=2\cos\left(\frac{3A+3B}{2}\right)\cos 90°=0$

28. $\cos^2 25+\cos^2 95+\cos^2 35$

$=\cos^2 35+\cos^2\left(60-35\right)+\cos^2\left(60+35\right)$

$=\frac{3}{2}$

29. $\frac{A}{2}=170^0\Rightarrow\frac{3\pi}{4}<\frac{A}{2}<\frac{5\pi}{4}$

30. $\cos\left(\alpha-\beta\right)=\frac{63}{65}$

$\cos^2\left(\frac{\alpha-\beta}{2}\right)=\frac{1+\cos\left(\alpha-\beta\right)}{2}=\frac{64}{65}$

NUMERICAL VALUE HINTS & SOLUTIONS

31. $\tan 9\theta = \frac{3}{4} \Rightarrow \sin 9\theta = \frac{3}{5}, \cos 9\theta = \frac{4}{5}$

Now $3\cos ec3\theta - 4\sec 3\theta = \frac{3\cos 3\theta - 4\sin 3\theta}{\sin 3\theta \cos 3\theta}$

$= \frac{2.5\left(\frac{3}{5}\cos 3\theta - \frac{4}{5}\sin 3\theta\right)}{2\sin 3\theta \cos 3\theta}$

$= \frac{10(\sin 9\theta \cos 3\theta - \cos 9\theta \sin 3\theta)}{\sin 6\theta}$

= 10

32. Put $t = \frac{\pi}{28}$

$\cos 2t \cos ec3t = \frac{\cos 2t}{\sin 3t} \times \frac{\sin t}{\sin t} = \frac{1}{2}\left(\frac{\sin 3t - \sin t}{\sin 3t \sin t}\right)$

$= \frac{1}{2}(\cos ect - \cos ec3t)$

Similarly $\cos 6t \cos ec9t = \frac{1}{2}(\cos ec3t - \cos ec9t)$ and $\cos 18t \cos ec27t = \frac{1}{2}(\cos ec9t - \cos ec27t)$

$\therefore \frac{1}{2}(\cos ect - \cos ec3t + \cos 3t - \cos ec9t + \cos ec9t - \cos ec27t)$

$= \frac{1}{2}\left(\cos ec\frac{\pi}{28} - \cos ec\frac{27\pi}{28}\right) = 0$

33. $(1+\cos 20)(1+\cos 60)(1+\cos 100)(1+\cos 140)$

$= 2\cos^2 20 \times \left(1+\frac{1}{2}\right)(2\cos^2 50)(2\cos^2 70)$

$= 8 \times \frac{3}{2} \times (\cos 10.\cos 50 \cos 70)^2$

$= 12\left(\frac{1}{4}\cos 30\right)^2 = \frac{9}{16}$

34. $\sin\theta, \cos\theta, \tan\theta$ are G.P

$\cos^2\theta = \sin\theta.\tan\theta = \frac{\sin^2\theta}{\cos\theta}$

$\cos^3\theta = 1 - \cos^2\theta$

$\cos^3\theta + \cos^2\theta = 1$

Cubing on both sides

$\cos^9\theta + \cos^6\theta + 3\cos^5\theta = 1$

35. $\left(\cos^6\frac{\pi}{16} + \sin^6\frac{\pi}{16}\right) + \left(\cos^6\frac{3\pi}{16} + \cos^6\frac{3\pi}{16}\right)$

$= 1 - 3\cos^2\left(\frac{\pi}{16}\right)\sin^2\frac{\pi}{16} + 1 - 3\cos^2\frac{3\pi}{16}\sin^2\frac{3\pi}{16}$

$= 2 - \frac{3}{4}\left(\frac{1-\cos\frac{\pi}{4} + 1 - \cos\frac{\pi}{4}}{2}\right) = \frac{5}{4}$

36. $\tan\frac{B}{2} = \frac{2\tan\frac{A}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}+\tan\frac{C}{2}}$

$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{A}{2}\tan\frac{C}{2} = 2\tan\frac{A}{2}\tan\frac{C}{2}$

$A+B+C=180 \Rightarrow \frac{A}{2}+\frac{B}{2}+\frac{C}{2}$

$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

$3\tan\frac{A}{2}\tan\frac{C}{2} = 1$

$\therefore \quad \cot\frac{A}{2}\cot\frac{C}{2} = 3$

37. $4\cos 36^0 + \cot 7\frac{1}{2}^0 = \sqrt{5}+1+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

$n_1 = 1,\ n_2 = 2, n_3 = 3,\ n_4 = 4,\ n_5 = 5,\ n_6 = 6$

$\sum_{i=1}^{6}(n_i)^2 = \frac{6.7.13}{6} = 91$

Product of the digits is $9\times 1 = 9$

38. $2\cos A + 9\cos D = -6\cos B - 7\cos C$ ……… (1)

$2\sin A - 9\sin D = -7\sin C + 6\sin B$ ………… (2)

$(1)^2 + (2)^2$

$\Rightarrow 85 + 36\cos(A+D) = 85 + 84\cos(B+C)$

$\Rightarrow \frac{\cos(A+D)}{\cos(B+C)} = \frac{7}{3}$

39. $\frac{1-\tan^2\theta}{1+\tan^2\theta} + \frac{\sin^2\alpha+1}{2} = \frac{-\sin^2\alpha+\sin^2\alpha+1}{2} = \frac{1}{2}$

40. $k = \sin\frac{\pi}{2^{10}}.\cos\frac{\pi}{2^{10}}.\cos 2\left(\frac{\pi}{2^{10}}\right).\cos 2^2\left(\frac{\pi}{2^{10}}\right)......\cos 2^8\left(\frac{\pi}{2^{10}}\right)$

$= \frac{\sin 2^9\left(\frac{\pi}{2^{10}}\right)}{2^9\sin\left(\frac{\pi}{2^{10}}\right)}.\sin\left(\frac{\pi}{2^{10}}\right) = \frac{1}{2^9}$

$\therefore \quad 2^{15}.k = 2^{15}.\frac{1}{2^9} = 2^6 = 64$

PERIODITY, MAX & MINI VALUES

1. $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

Period = 2π

2. $\sin x\cos x = \frac{1}{2}\sin 2x$

Period = $\frac{2\pi}{2} = \pi$

3. Minimum value $= -\sqrt{16+12}$

$=-\sqrt{28}=-2\sqrt{7}$

4. $-1\le \cos\sqrt{3+x+x^2}\le 1$

$\Rightarrow 3\ge -3\cos\sqrt{3+x+x^2}\ge -3$

$\therefore$ $f(x)$ lies in the interval [-3,3]

5. Minimum $=-\sqrt{1+8}=-3$

Maximum $=\sqrt{1+8}=3$

$\therefore -3\le A\le 3$

6. Period of $\sin\frac{\pi x}{2}=\frac{2\pi}{\frac{\pi}{2}}=4$

Period of $\cos\frac{\pi x}{3}=\frac{2\pi}{\frac{\pi}{3}}=6$

$\therefore$ L.C.M {4,6} = 12

7. Minimum value $=-\sqrt{64+225}=-17$

8. Minimum value $=2\sqrt{4\times 9}=12$

TRIGNOMETRIC EQUATIONS

1. $(x-1)^2+3+3\sin(ax+b)=0$

$(x-1)^2+3=-3\sin(ax+b)$

$L.H.S\ge 3, RHS\in[-3,3]$ now

$\sin(ax+b)=-1$

$\therefore x=1\ \sin(b+a)=-1$

2. Given, $1+\sin\theta+\sin^2\theta+\sin^3\theta+....\infty=4+2\sqrt{3}$

$\Rightarrow \frac{1}{1-\sin\theta}=4+2\sqrt{3}\,[\because 0<\sin\theta<1]$

$\Rightarrow 1-\sin\theta=\frac{4-2\sqrt{3}}{16-12}=1-\frac{\sqrt{3}}{2}$

$\Rightarrow \sin\theta=\frac{\sqrt{3}}{2}$

$\Rightarrow \theta=\frac{\pi}{3}\,or\,\frac{2\pi}{3}$

3. $\left\{\theta:\theta=2n\pi\pm\left(\frac{\pi}{3}\right), n\in I\right\}$

$\Rightarrow \cos^2\theta=\left(\frac{1}{6}\sin\theta\right)(\tan\theta)$

$\Rightarrow \cos^2\theta=\frac{1}{6}\left(\frac{\sin^2\theta}{\cos\theta}\right)$

$\Rightarrow 6\cos^3\theta=1-\cos^2\theta$

$\Rightarrow 6\cos^3\theta+\cos^2\theta-1=0$

$\Rightarrow \cos\theta=\frac{1}{2}$

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

4. $\tan(k+1)\theta = \tan\theta$ is given by

$(k+1)\theta = n\pi + \theta \Rightarrow k\theta = n\pi, n \in I$

$\therefore \theta \in \frac{n\pi}{k} : n \in I$

5. $\sec 2\theta = \frac{1}{\cos 2\theta} = \frac{1+\tan^2\theta}{1-\tan^2\theta}$

$\Rightarrow$ We can write the given equation as

$\tan^2\theta + \frac{1+\tan^2\theta}{1-\tan^2\theta} = 1$

$\Rightarrow \tan^2\theta\left(1-\tan^2\theta\right) + 1 + \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow 3\tan^2\theta - \tan^4\theta = 0$

$\Rightarrow \tan^2\theta\left(3-\tan^2\theta\right) = 0$

$\Rightarrow \tan\theta = 0 \text{ or } \tan^2\theta = 3 = \left(\sqrt{3}\right)^2$

$\Rightarrow \tan\theta = 0 \Rightarrow \theta = m\pi, m \in I$

or $\tan^2\theta = \tan^2\frac{\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$

6. $\sin x + \sin\left(120^0 + x\right) + \sin\left(240^0 + x\right) = 0$

$\Rightarrow \sin^3 x + \sin^3\left(120^0 + x\right) + \sin^3\left(240^0 + x\right)$

$= 3\sin x \sin\left(120 + x\right)\sin\left(240 + x\right)$

$= 3\left(-\frac{1}{4}\right)\sin 3x \Rightarrow \sin 3x = \cos 2x$

$\Rightarrow \cos\left(\frac{\pi}{2} - 3x\right) = \cos 2x$

$\Rightarrow 2x = 2k\pi \pm \left(\frac{\pi}{2} - 3x\right)$

$\Rightarrow 5x = 2k\pi + \frac{\pi}{2} = \left(4k+1\right)\frac{\pi}{2} \forall k \in Z$

or $x = -2k\pi + \frac{\pi}{2}, \forall k \in Z$

$\Rightarrow x = \left(4k+1\right)\frac{\pi}{10}, \forall k \in Z \left\{\cos x \neq 0\right\}$

7. Given, $2\cos^2 x - 1 + 2\cos^2 x = 2$

$\Rightarrow \cos^2 x = \frac{3}{4}$

$\cos^2 x = \cos^2\frac{\pi}{6}$

$\therefore\ x = n\pi \pm \frac{\pi}{6} : n \in Z$

8. $\frac{\left(1+\tan x+\tan^2 x\right)\left(1+\tan^2 x-\tan x\right)}{\tan^2 x} > 0$

$\Rightarrow \dfrac{\left(1+\tan^2 x\right)^2-\tan^2 x}{\tan^2 x}>0$

Since, $1+\tan^2 x>\tan^2 x$,

$\forall x \in R-\left\{x=\dfrac{n\pi}{2}, n\in I\right\}$

Hence, given expression is positive for all values of

$x\in R-\left\{x=\dfrac{n\pi}{2}, n\in I\right\}\infty$

9. $\because 5+8\sin^2 x\in[5,13]$

Also, $2y^2-8y+21=2(y-2)^2+13\geq 13$

So, equality should hold true if,

$5+8\sin^2 x=13$ and $2y^2-8y+21=13$

$\Rightarrow \sin x=\pm 1,\ y=2$

$\Rightarrow \sin x=(2n+1)\dfrac{\pi}{2},\ y=2$

For least value, $x=\pm\dfrac{\pi}{2}$

Hence, $x^2y^3=\dfrac{\pi^2}{4}.8=2\pi^2$

10. Given, $4(\cos x-\sin x)=-5$

$4\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}-\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)=-5$

$\Rightarrow 4-4\tan^2\dfrac{x}{2}-8\tan\dfrac{x}{2}=-5-5\tan^2\dfrac{x}{2}$

$\Rightarrow \tan^2\dfrac{x}{2}-8\tan\dfrac{x}{2}+9=0$

$\Rightarrow 9\cot^2\dfrac{x}{2}-8\cot\dfrac{x}{2}+1=0$

$\Rightarrow \cot\dfrac{x}{2}=\dfrac{4\pm\sqrt{7}}{9}$

Since, $\dfrac{\pi}{4}<\dfrac{x}{2}<\dfrac{3\pi}{8}\Rightarrow \sqrt{2}-1<\cot\dfrac{x}{2}<1$

$\Rightarrow \cot\dfrac{x}{2}=\dfrac{4+\sqrt{7}}{9}$

11. If x lies in 1st or 3rd quadrant then $|\cot x|=\cot x$

thus equation becomes $\cot x=\cot x+\dfrac{1}{\sin x}\Rightarrow\dfrac{1}{\sin x}=0$, not possible

If x lies in 2nd or 4th quadrant, then $|\cot x|=-\cot x$,

thus equation becomes $-\cot x=\cot x+\dfrac{1}{\sin x}$

$\Rightarrow -2\cot x=\dfrac{1}{\sin x}$

$\Rightarrow -2\cos x\sin x=\sin x$

$\Rightarrow \sin x(1+2\cos x)=0 \Rightarrow \cos x=-\frac{1}{2}$

$(\because \sin x \neq 0)$

$\Rightarrow x=\frac{2x}{3}$ is the only solution

(In 2nd quad.)

In $[0,2\pi]$ there is only one solution

In $[0,10\pi]$, we get, five solutions

12. $\cos 2x + a\sin x = 2a-7$

$i.e.\, 2\sin^2 x - a\sin x + 2a - 8 = 0$

$\sin x = \frac{a \pm \sqrt{a^2-8(2a-8)}}{4} = \frac{a \pm (a-8)}{4}$

$\sin x = \frac{a-4}{2}$ *or* 2

Hence, $-1 \le \frac{(a-4)}{2} \le 1$

The range of a is [2,6]

13. Since, $\sin x > 0, \cos x > 0, \cos x \neq 1$

So, $\sin x \le \cos^2 x$

$\Rightarrow \sin^2 x + \sin x - 1 \le 0$

$\Rightarrow \left(\sin x + \frac{1}{2}\right)^2 - \frac{5}{4} \le 0$

$\Rightarrow \left|\sin x + \frac{1}{2}\right| \le \frac{\sqrt{5}}{2}$

$\therefore \sin x + \frac{1}{2} \in \left[\frac{-\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$

(As, $\sin x > 0$)

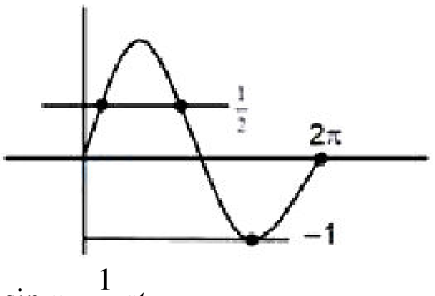
$\therefore 0 < \sin x \le \frac{\sqrt{5}-1}{2}$

14. $\tan x + \sec x = 2\cos x$

Multiplying by $\cos x$

$\sin x + 1 = 2\cos^2 x = 2 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow \sin x = -1, \frac{1}{2}$



$\sin x = \frac{1}{2} at$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ in $[0, 2\pi]$

$\sin x = \frac{1}{2} \to$ four times in

$\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$ for which and $\sec x$ are not defined.

$\Rightarrow \sin x = -1$ is not possible

$\Rightarrow$ Number of roots in $[0, 4\pi]$ is 4.

15. $\cos^2\theta - 3\cos\theta + 2 \geq \frac{1}{\cos ec^2\theta} = \sin^2\theta$

$\cos^2\theta - 3\cos\theta + 2 \geq 1 - \cos^2\theta$

$2\cos^2\theta - 3\cos\theta + 1 \geq 0$

$(2\cos\theta - 1)(\cos\theta - 1) \geq 0$

$\Rightarrow \cos\theta \leq \frac{1}{2}\ or\ \cos\theta \geq 1$

$\theta \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ of the given intervals.

16. $f(\sin^2\theta) + f(1 - \sin^2\theta)$

Let, $\sin^2\theta = t$

$\therefore f(t) + f(1-t)$

$= \frac{25^t}{25^t + 5} + \frac{25^{1-t}}{25^{1-t} + 5} = \frac{25^t}{25^t + 5} + \frac{25}{25 + 5(25^t)}$

$= \frac{25^t + 5}{25^t + 5} = 1$

Therefore, the given equation will become $\tan^2\theta = 1$

$\therefore \tan\theta = \pm 1 \Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in Z$

Hence, the number of solutions are $4 \times 5 = 20$

$\{\because 4 \text{ solutions in } [0, 2\pi]\}$

17. $(2\cos^2 2x - 1) + 6 = 7\cos 2x$

On putting $\cos 2x = t$, we get,

$2t^2 - 1 + 6 = 7t$

$2t^2 - 7t + 5 = 0$

$(2t - 5)(t - 1) = 0$

$t = \frac{5}{2}, 1$

$T = \frac{5}{2}$ (not possible)

$t = 1 \Rightarrow \cos 2x = 1 \Rightarrow 2x = 2n\pi$

$\Rightarrow x = n\pi$

The roots in [0,314] are

$\pi, 2\pi, 3\pi, ..., 99\pi\ \{100\pi > 314\}$

Sum of roots

$= \pi + 2\pi + 3\pi + ... + 99\pi = 4950\pi$

$\Rightarrow \lambda = 4950$

18. $\sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\cos x = -\frac{\sqrt{3}}{2} \Rightarrow x = \frac{5\pi}{6}, \frac{7\pi}{6}$

$\Rightarrow \alpha = \frac{7\pi}{6}, \beta = \frac{11\pi}{6}, \gamma = \frac{5\pi}{6}$

$\Rightarrow \alpha + \beta = \frac{7\pi}{6} + \frac{11\pi}{6}$

$= 3\pi, \beta - \gamma = \frac{11\pi}{6} - \frac{5\pi}{6} = \pi$

$\Rightarrow \frac{\alpha+\beta}{|\beta-\gamma|} = \frac{3\pi}{\pi} = 3$

19. $\cot x = -\sqrt{3}$

$\Rightarrow x = \frac{5\pi}{6}, \frac{11\pi}{6}$ and $\cos ecx = -2$

$\Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$

So, $\alpha = \frac{11\pi}{6}, \beta = \frac{5\pi}{6}, \gamma = \frac{7\pi}{6}$

$\Rightarrow \alpha - \beta = \pi$ and $\beta + \gamma = 2\pi$

$\Rightarrow \frac{|\alpha-\beta|}{\beta+\gamma} = \frac{\pi}{2\pi} = \frac{1}{2}$

NUMERICAL

20. Given equation is $1 - \sin^2 x + \frac{\sqrt{3}+1}{2}\sin x - \frac{\sqrt{3}}{4} - 1 = 0$

$\Rightarrow \sin^2 x - \frac{\sqrt{3}+1}{2}\sin x + \frac{\sqrt{3}}{4} = 0;$

$\Rightarrow 4\sin^2 x - 2\sqrt{3}\sin x - 2\sin x + \sqrt{3} = 0$

$2\sin x\left(2\sin x - \sqrt{3}\right) - \left(2\sin x - \sqrt{3}\right) = 0$

$\Rightarrow (2\sin x - 1)\left(2\sin x - \sqrt{3}\right) = 0$

On solving we get $\sin x = \frac{1}{2}; \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}; \frac{\pi}{3}, \frac{2\pi}{3}$

21. $\sin^4 x + \cos^4 x = \sin x \cos x$

$\Rightarrow \left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x\cos^2 x = \sin x\cos x$

$\Rightarrow 2\sin^2 x\cos^2 x + \sin x\cos x - 1 = 0$

$\Rightarrow (2\sin x\cos x - 1)(\sin x\cos x + 1) = 0$

$\Rightarrow \sin 2x = 1....(i)$

$x \in [\pi, 5\pi] \Rightarrow 2x \in [2\pi, 10\pi]$

From (i), $2x = \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$

$\Rightarrow x = \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}$

$\Rightarrow$ The number of solutions $= 4$

22. Frist equation is $4\sin^2\theta\cos^2\theta = 2\cos^2\theta$

$\Rightarrow \cos^2\theta = 0 \; or \; \sin^2\theta = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2 = \sin^2\frac{\pi}{2}$

$\Rightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I \; or \; \theta = n\pi \pm \frac{\pi}{4}, \; n \in I$

Second equation is not satisfied by

$\theta = (2n+1)\frac{\pi}{2}, n \in I$ but satisfied by

$\theta = n\pi \pm \frac{\pi}{4}, n \in I$

$\frac{-5\pi}{4}, \frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

23. In the interval $[-\pi, \pi]$ all the values of $\sin|x|$ are positive as well as $|\cos x|$

Hence in $[-\pi, \pi]$, equation gets reduced to, $2^{\sin x} = 3^{\cos x}$

Taking log on both sides, $\sin x \log 2 = \cos x \log 3$

$\Rightarrow |\tan x| = \log 3 / \log 2$, value of are positive in 1st and 3rd quadrant and negative in 2nd and 4th quadrant but positive for $|\tan x|$ in all the values of x in $[-\pi, \pi]$.

Hence, $\tan x$ will repeat its value in all 4 quadrants, so the number of solutions are 4.

24. $\log_{\sqrt{2}\sin x}(1+\cos x) = 2 ...(i)$

$\sqrt{2}\sin x \neq 1, \sqrt{2}\sin x > 0, 1+\cos x > 0$

$\Rightarrow x \in (0, \pi) - \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\} \Rightarrow \sin x \neq \frac{1}{\sqrt{2}}, \sin x > 0$ and $x \neq 0$ or a multiple of π

(feasible region)

Hence, from (1), we get, $\left(\sqrt{2}\sin x\right)^2 = 1 + \cos x$

$\Rightarrow 2\sin^2 x = 1 + \cos x$

$\Rightarrow 2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{1}{2} ...... [\cos x + 1 > 0]$

$\Rightarrow x = \frac{\pi}{3} \Rightarrow p = 1, q = 3 \Rightarrow p^2 + q^2 = 10$

25. $2\sin x = 5x^2 + 2x + 3$

$\Rightarrow 2\sin x = 4x^2 + (x+1)^2 + 2$

But, $2\sin x \leq 2$

and $4x^2 + (x+1)^2 + 2 > 2$, so it has no solution

26. $\pi \log_3\left(\frac{1}{x}\right) = k\pi, k \in I$

$\log_3\left(\frac{1}{x}\right) = k \Rightarrow x = 3^{-k}$

Possible values of k are -1, 0,2,3,.......

$S = (3+1) + \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...\infty\right)$

$$= 4 + \frac{(1/3)}{1-(1/3)} = 4 + \frac{1}{2} = \frac{9}{2}$$

27. *Let* $81^{\sin^2 x} = t,$

So, the given equation is $t + \frac{81}{t} = 30$

$t^2 - 30t + 81 = 0 \Rightarrow t = 3 \; or \; 27$

$81^{\sin^2 x} = 3 \; or \; 27$

$\Rightarrow 3^{4\sin^2 x} = 3^1 \; or \; 3^3$

$\Rightarrow \sin^2 x = \frac{1}{4} \; or \; \frac{3}{4} \Rightarrow \sin x = \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$

Hence, there are 8 solutions between 0 and 2π

28. $\sqrt{3}\cos x - \sin x = \pm 2$

$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \pm 1$

$\cos\left(x + \frac{\pi}{6}\right) = \pm 1$

$\Rightarrow x + \frac{\pi}{6} = n\pi$

$\Rightarrow x = n\pi - \frac{\pi}{6}$

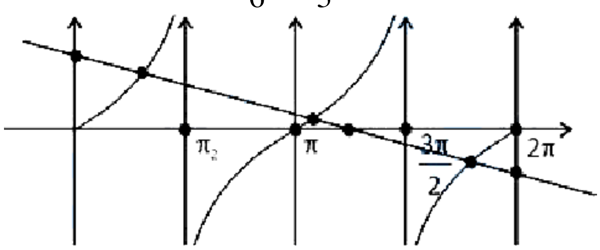
$\Rightarrow x = \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$

$\Rightarrow$ The sum of roots is $10\pi - \frac{2\pi}{3} = \frac{28}{3}\pi$

$\Rightarrow 6k = 6 \times \frac{28}{3} = 56$

29. $2x + 3\tan x = \frac{5\pi}{2} \Rightarrow \tan x = -\frac{2}{3}x + \frac{5\pi}{6}$

$y = \tan x \; and \; y = \frac{5\pi}{6} - \frac{2x}{3}$



Both the graphs meet exactly three times in $[0, 2\pi]$.

Thus, there are 3 solutions.

30. $y = \cos x; y = \sin 3x$

$\Rightarrow \cos x = \cos\left(\frac{\pi}{2} - 3x\right)$

$\Rightarrow x = 2n\pi \pm \left(\frac{\pi}{2} - 3x\right)$

$\Rightarrow 4x = 2n\pi + \frac{\pi}{2}, -2x = 2n\pi - \frac{\pi}{2}$

$\Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}, x = k\pi + \frac{\pi}{4}$

$\Rightarrow x = (4n+1)\frac{\pi}{8}, x = (4k+1)\frac{\pi}{4}$

$n = 0 \Rightarrow x = \frac{\pi}{8}, k = 0 \Rightarrow x = \frac{\pi}{4}$

$n = -1 \Rightarrow x = \frac{-3\pi}{8}$

$\Rightarrow x = \frac{-3\pi}{8}, \frac{\pi}{8}$ and $\frac{\pi}{4}$

31. $LHS = 12\sin x + 5\cos x \in \left[-\sqrt{12^2+5^2}, \sqrt{12^2+5^2}\right]$

i.e. [-13,13] i.e.
maximum value of LHS is 13

$RHS = 2(y^2 - 4y + 4) + 13$

$= 2(y-2)^2 + 13$

$RHS \geq 13$

Roots of the equation exist if
LHS = RHS=13
RHS = 13 when y = 2

$LHS = 13 \Rightarrow 12\sin x + 5\cos x = 13$

$\Rightarrow \frac{12}{13}\sin x + \frac{5}{13}\cos x = 1$

$\sin(x+\alpha) = 1$, where $\tan\alpha = \frac{5}{12}$

$x + \tan^{-1}\frac{5}{12} = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2} - \tan^{-1}\frac{5}{12}$

$\Rightarrow xy = \pi - 2\tan^{-1}\frac{5}{12}$

$\Rightarrow \frac{5}{12} = \tan\left(\frac{\pi}{2} - \frac{xy}{2}\right) = \cot\left(\frac{xy}{2}\right) \quad \Rightarrow 12\cot\left(\frac{xy}{2}\right) = 5$

INVERSE TRIG. FUNCTIONS

1. Given that, $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3} = \sin^{-1} x$

Taking sine on both sides

$\Rightarrow \left(\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right) = x$

$\Rightarrow \left(\frac{1}{3}.\frac{\sqrt{5}}{3} + \frac{2}{3}.\frac{\sqrt{8}}{3}\right) = x$

$\Rightarrow \left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right) = x$

$\therefore x = \left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right)$

2. Let $x = -y, y > 0$

$\therefore \sin^{-1} x = \sin^{-1}(-y)$

$= -\sin^{-1} y$

$= -\cos^{-1}\sqrt{1-y^2}$

$= -\cos^{-1}\sqrt{1-x^2}$

3. $\alpha + \beta = \sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{3} + \cos^{-1}\frac{1}{3}$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Also, $\alpha = \frac{\pi}{3} + \sin^{-1}\frac{1}{3} < \frac{\pi}{3} + \sin^{-1}\frac{1}{2}$

As $\sin\theta$ is increasing in $\left[0, \frac{\pi}{2}\right]$

$\therefore \alpha < \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

$\Rightarrow \beta > \frac{\pi}{2} > \alpha$

$\Rightarrow \alpha < \beta$

4. As, $\cos^{-1}(\cos 4) = \cos^{-1}\{\cos(2\pi - 4)\} = 2\pi - 4$

$\Rightarrow 2\pi - 4 > 3x^2 - 4x$

$\Rightarrow 3x^2 - 4x - (2\pi - 4) < 0$

$\Rightarrow \frac{2-\sqrt{6\pi - 8}}{3} < x < \frac{2+\sqrt{6\pi-8}}{3}$

5. Let, $\cot^{-1} x = A, (\cot^2 A) = x^2$

$1 + (\cot^2 A) = 1 + x^2 = \cos ec^2 A$

$\frac{1}{\sin^2 A} = 1 + x^2 \Rightarrow \sin A = 1/\sqrt{1+x^2}$

$\Rightarrow \sin(\cot^{-1} x) = 1/\sqrt{1+x^2}$

6. $\because [\sin^{-1} x] > [\cos^{-1} x] \Rightarrow x > 0$

Hence, $[\cos^{-1} x] = \begin{cases} 0, & x \in (\cos 1, 1] \\ 1, & x \in (0, \cos 1] \end{cases}$

and $[\sin^{-1} x] = \begin{cases} 0, & x \in (0, \sin 1) \\ 1, & x \in [\sin 1, 1] \end{cases}$

$\therefore x \in [\sin 1, 1]$

7. Given, $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$

$\text{RHS} = \frac{\pi}{4} + \tan^{-1} x = \tan^{-1} 1 + \tan^{-1} x$

$= \tan^{-1}\left(\frac{1+x}{1-x}\right),$

$\therefore x \in (-\infty, 1)$

8. Since, $\tan^{-1} x$ and $\cot^{-1} x$ exists for all $x \in R$ and $\cos^{-1}(2-x)$ exist if $-1 \le 2 - x \le 1$

$\Rightarrow 1 \le x \le 3$

Thus, the domain of the given function is [1,3]

9. Given expression $= \tan^{-1} 2 + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{2}{25} + \tan^{-1}\frac{2}{49} + ...$

General term $= \frac{2}{(2n-1)^2} = \frac{2}{4n^2-4n+1}$

$= \frac{2}{1+4n(n-1)} = \frac{2n-(2n-2)}{1+2n(2n-2)}$

$T_n = \tan^{-1} 2n - \tan^{-1}(2n-2)$

$\therefore$ Sum of the series

$= \tan^{-1} 2 - \tan^{-1} 0 + \tan^{-1} 4 - \tan^{-1} 2 + \tan^{-1} 6 - \tan^{-1} 4 + ... \tan^{-1} 2n - \tan^{-1}(2n-2)$

$= \tan^{-1} 2n - \tan^{-1} 0 = \tan^{-1} 2n$

10. Here, $x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1$

But, $-1 \leq (x^2 - 2x + 2) \leq 1$

Which is possible only when

$x^2 - 2x + 2 = 1$

$\Rightarrow x = 1$

Then, $a(1)^2 + \sin^{-1}(1) + \cos^{-1}(1) = 0$

$\Rightarrow a + \frac{\pi}{2} + 0 = 0$

$\Rightarrow a = -\frac{\pi}{2}$

11. $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{x^2+x} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

This equation holds true, if

$x^2 + x \geq 0$ and $0 \leq x^2 + x + 1 \leq 1$

Now, $x^2 + x \geq 0$ and $0 \leq x^2 + x + 1 \leq 1$

$\Rightarrow x^2 + x \geq 0$ and $x^2 + x + 1 \leq 1$

$[\because x^2 + x + 1 > 0 \text{ for all } x]$

$\Rightarrow x^2 + x \geq 0$ and $x^2 + x \leq 0$

$\Rightarrow x^2 + x = 0 \Rightarrow x = 0, -1$

Clearly, these two values satisfy the given equation, Hence, $x = -1, 0$ are the solutions of the given equation.

12. Since, x, y, z are in AP

$\therefore y = \frac{x+z}{2}$ .... (i)

And $\tan^{-1} x, \tan^{-1} y$ and $\tan^{-1} z$ are also in AP.

$\therefore 2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$

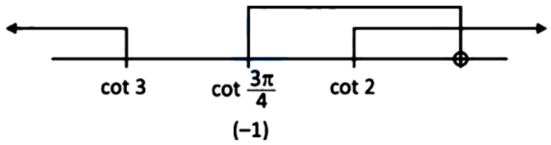
$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow \frac{2y}{1-y^2} = \frac{2y}{1-xz}$ [from equation. (i)]

$\Rightarrow y^2 = xz$

$\Rightarrow x, y, z$ are in GP.
$\therefore x = y = z$

13. $\cos^{-1} x > \cos^{-1} x^2 \Rightarrow x < x^2 \;\&\; x \in [-1,1]$
$\{\because \cos^{-1} x$ is a decreasing function$\}$
$\Rightarrow x(x-1) > 0 \;\&\; x \in [-1,1]$
$\Rightarrow x \in (-\infty, 0) \cup [1, \infty) \;\&\; x \in [-1,1]$
$\Rightarrow x \in [-1, 0)$ ....(1)
Now $\left(\cot^{-1} x\right)^2 - 5\cot^{-1} x + 6 > 0$
$\Rightarrow \left(\cot^{-1} x - 2\right)\left(\cot^{-1} x - 3\right) > 0$
$\Rightarrow \cot^{-1} x < 2$ or $x < \cot 3$ ....(2)
$\{\because \cot^{-1} x$ is a decreasing function$\}$



$(1) \cap (2) \Rightarrow [\cot 2, 0)$

14. For $f(x)$ to be defined $\left[2^x\right] = 0, 1$
$\therefore 2^x \in [0, 2) \Rightarrow x \in (-\infty, 1)$
Range $\left\{\cos^{-1}(0) + \sin^{-1}(-1), \cos^{-1}(1) + \sin^{-1}(0)\right\} \equiv \{0\}$
$\therefore f(x) = 0 \forall x \in (-\infty, 1)$
So, Domain is not symmetric about the origin
$\therefore f(x)$ is neither even nor odd

15. $\left(\cot^{-1} x - 5\right)\left(\cot^{-1} x - 2\right) > 0$
$\Rightarrow \cot^{-1} x \in (-\infty, 2) \cup (5, \infty)$
$\Rightarrow \cot^{-1} x \in (0, 2)$ Taking intersection with range of $\cot^{-1} x$)
$\Rightarrow x \in (\cot 2, \infty)$

16. $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2} \in [0, 1)$
$\therefore$ Range of $f(x) \in \left[\sin^{-1}(0), \sin^{-1}(1)\right)$
$\in \left[0, \dfrac{\pi}{2}\right)$

17. Let, $\tan^{-1} x = t$
the given inequality becomes
$t^2 + 3t - 4 > 0 \Rightarrow (t+4)(t-1) > 0$
$\therefore t < -4$ or $t > 1$
$\tan^{-1} x < -4$ (not possible) or $\tan^{-1} x > 1$
$\Rightarrow x > \tan 1$

18. Since, a_1, a_2, a_3 are in A.P.
$\Rightarrow a_2 - a_1 = d = a_3 - a_2$
Now, $\tan^{-1}\left(\dfrac{d}{1+a_1 a_2}\right) + \tan^{-1}\left(\dfrac{d}{1+a_2 a_3}\right)$

$= \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_2 a_3}\right)$

$\left(\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x - y}{1 + xy}\right)$

$= \tan^{-1} a_2 - \tan^{-1} a_1 + \tan^{-1} a_3 - \tan^{-1} a_2$

$= \tan^{-1} a_3 - \tan^{-1} a_1$

$= \tan^{-1}\left(\frac{a_3 - a_1}{1 + a_1 a_3}\right)$

$= \tan^{-1}\left(\frac{(a_3 - a_2) + (a_2 - a_1)}{1 + a_1 a_3}\right)$

$= \tan^{-1}\left(\frac{2d}{1 + a_1 a_3}\right)$

19. $\because x^2 + 4x \geq 0$ for the first term

and $0 \leq x^2 + 4x + 1 \leq 1$ for the second term

$\therefore$ by both the results, there is only one possibility

$x^2 + 4x = 0$

$\Rightarrow x = 0, -4$

$\therefore f(x) = \tan^{-1}(0) + \sin^{-1}(1) = \frac{\pi}{2}$

20. $f(x) = \sin^{-1}\left(x\sqrt{1 - (\sqrt{x})^2} - \sqrt{x}\sqrt{1 - x^2}\right)$

$\because \sin^{-1}(a) - \sin^{-1}(b) = \sin^{-1}\left(a\sqrt{1 - b^2} - b\sqrt{1 - a^2}\right)$

$\Rightarrow f(x) = \sin^{-1} x - \sin^{-1}\sqrt{x} \leq 0$

21. $y = \tan^{-1}\left(\frac{2^{x+1} - 2^x}{1 + 2^x.2^{x+1}}\right) = \tan^{-1} 2^{(x+1)} - \tan^{-1} 2^x$

$\sum_{r=0}^{9} f(r) = \tan^{-1}(2^{10}) - \tan^{-1} 1 = \tan^{-1}\left(\frac{1023}{1025}\right)$

22. Let, $x = \sin 2\theta$ (where $\tan\theta = 3$)

$\Rightarrow x = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{6}{1 + 9} = \frac{3}{5}$

If $\alpha = \tan^{-1}\frac{4}{3}$

$\Rightarrow y = \sin\left(\frac{\alpha}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1 - \cos\alpha}$

$= \frac{1}{\sqrt{2}}\sqrt{1 - \frac{3}{5}} = \frac{1}{\sqrt{5}}$

$\therefore y^2 = \frac{1}{5} \Rightarrow y^2 = 2x - 1$

23. $\angle A = \angle B = \sin^{-1}\left(\frac{\sqrt{6} + 1}{2\sqrt{3}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \tan^{-1}\left(\frac{\sqrt{6} + 1}{\sqrt{3} - \sqrt{2}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$$= \tan^{-1}\left(\frac{\frac{\sqrt{6}+1}{\sqrt{3}-\sqrt{2}}+\frac{1}{\sqrt{2}}}{1-\frac{\sqrt{6}+1}{\sqrt{3}-\sqrt{2}}.\frac{1}{\sqrt{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\left(\sqrt{6}+1\right)\sqrt{2}+\left(\sqrt{3}-\sqrt{2}\right)}{\sqrt{2}\left(\sqrt{3}-\sqrt{2}\right)-\left(\sqrt{6}+1\right)}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{12}+\sqrt{2}+\sqrt{3}-\sqrt{2}}{\sqrt{6}-2-\sqrt{6}-1}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{3}+\sqrt{3}}{-3}\right) = \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right)$$

$$= \tan^{-1}\left(-\sqrt{3}\right) = \frac{2\pi}{3}$$

According to question

$$\angle A = \angle B = \frac{1}{2}\left(\sin^{-1}\left(\frac{\sqrt{6}+1}{2+\sqrt{3}}\right)+\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$\Rightarrow \angle A = \angle B = \frac{1}{2}\left(\frac{2\pi}{3}\right)$$

$$\Rightarrow \angle A = \angle B = \frac{\pi}{3}$$

$$\therefore \angle C = \pi - \angle A - \angle B$$

$$= \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

Since ΔABC is an equilateral Δ

$$\therefore \text{area}\left(\Delta ABC\right) = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}c^2$$

$$= \frac{\sqrt{3}}{4}.\left(6.3^{\frac{1}{4}}\right)^2$$

$$= \frac{\sqrt{3}}{4}\sqrt{3}.6\times 6$$

$$3\times 3\times 3 = 27$$

24. $x+\dfrac{1}{x} = 2 \Rightarrow x = 1....$

$$\left[x+\frac{1}{x} \geq 2 \text{ as equality occurs only when } x = 1\right]$$

$$\Rightarrow \sin^{-1} x = \frac{\pi}{2}$$

Similarly, $y = -1....$

$$\left[y+\frac{1}{y} \leq -2 \text{ as equality occurs only when } y = -1\right]$$

$$\Rightarrow \cos^{-1} y = \pi$$

$$\sin^{-1} x + \cos^{-1} y = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

$\Rightarrow m = \frac{3}{2} = 1.5$

25. We have, $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$

$\Rightarrow 1-x = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right)$

$\Rightarrow 1-x = \cos\left(2\sin^{-1}x\right)$

$\Rightarrow 1-x = \cos\left\{\cos^{-1}\left(1-2x^2\right)\right\}$

$\left[\because 2\sin^{-1}x = \cos^{-1}\left(1-2x^2\right)\right]$

$\Rightarrow 1-x = \left(1-2x^2\right)$

$\Rightarrow x = 2x^2 \Rightarrow x(2x-1) = 0 \Rightarrow x = 0, \frac{1}{2}$

For, $x = \frac{1}{2}$, we have

LHS $= \sin^{-1}(1-x) - 2\sin^{-1}x$

$= \sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{2} = -\sin^{-1}\frac{1}{2} = \frac{-\pi}{6} \neq \text{RHS}$

So, $x = \frac{1}{2}$ is not a root of the given equation.

Clearly, $x = 0$ satisfies the equation

Here, $x = 0$ is the root of the given equation

26. Let, $\cot^{-1}(x+1) = \theta$ and $\tan^{-1}x = \phi$

Now, given equation becomes $\sin\theta = \cos\phi$

$\Rightarrow \frac{1}{\sqrt{1+(x+1)^2}} = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow (x+1)^2 + 1 = x^2 + 1 \Rightarrow x = -\frac{1}{2}$

27. We know that, $\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x}$

$\therefore \tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right) = \frac{1-\cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)}$

$= \frac{1-2/\sqrt{5}}{\sin\left(\sin^{-1}\left(1/\sqrt{5}\right)\right)}$

$= \frac{1-2/\sqrt{5}}{1/\sqrt{5}} = \sqrt{5} - 2$

Now, $a + \sqrt{b} = -2 + \sqrt{5}$

$\Rightarrow a = -2$ and b = 5

Hence, $\frac{a+b}{b} = \frac{3}{8} = 0.6$

28. $f(x)=\cos^{-1}\left\{x\sqrt{x}-\sqrt{(1-x)(1-x^2)}\right\}$

$=\cos^{-1}\left\{x\sqrt{x}-\sqrt{1-(\sqrt{x})^2}\sqrt{1-x^2}\right\}$

$=\cos^{-1}x+\cos^{-1}\sqrt{x}$

At $x=1$

minimum value of $f(x)=0$

29. We have, $\frac{2\pi}{3}<\cot^{-1}\left(\frac{n}{2\pi}\right)<\pi\Rightarrow\cot\frac{2\pi}{3}>\frac{n}{2\pi}>-\infty$

$\Rightarrow-\frac{1}{\sqrt{3}}>\frac{n}{2\pi}>-\infty\Rightarrow-\frac{2\pi}{\sqrt{3}}>n>-\infty$

Maximum value of n is -4

30. $\because\sin^{-1}\left(\frac{2\sqrt{15}}{|x|}\right)=\cos^{-1}\sqrt{1-\left(\frac{2\sqrt{15}}{|x|}\right)^2}$ for

$0<\frac{2\sqrt{15}}{|x|}\le1\Rightarrow|x|\ge2\sqrt{15}$

$\therefore\left(\frac{14}{|x|}\right)^2=1-\left(\frac{2\sqrt{15}}{|x|}\right)^2\Rightarrow|x|=16\ \Rightarrow x\pm16$

which satisfies $|x|\ge2\sqrt{15}$

Hence, the maximum value of x = 16

31. $y=\cos^{-1}\left(\frac{7}{2}(1+\cos2x)+\sqrt{(\sin^2x-48\cos^2x)}\sin x\right)$

$=\cos^{-1}\left((7\cos x)(\cos x)+\sqrt{1-49\cos^2x}\sqrt{1-\cos^2x}\right)$

$=\cos^{-1}(\cos x)-\cos^{-1}(7\cos x)$

$=x-\cos^{-1}(7\cos x)$

Hence, the value of k = 7

PROPERTIES OF TRIANGLES

1. $\frac{a^2+b^2}{a^2-b^2}\sin(A-B)=1$

$\Rightarrow\sin^2A+\sin^2B=\sin(A+B)$

$\Rightarrow\sin^2A+1-\cos^2B=\sin C$

$\Rightarrow\cos(B+A)\cos(B-A)=1-\cos\left(\frac{\pi}{2}-C\right)$

$\Rightarrow\cos(B-A)=\frac{2\sin^2\left(\frac{\pi}{4}-\frac{C}{2}\right)}{\sin\left(\frac{\pi}{2}-C\right)}$

$=tan\left(\frac{C}{2}-\frac{\pi}{4}\right)$

2. $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$

$\Rightarrow\cot A=\cot B=\cot C$

$\Rightarrow A=B=C$

3. $\Rightarrow \sin A+\sin B=\frac{\sqrt{6}}{2}$, $\sin A\sin B=\frac{1}{4}$

$\Rightarrow \sin A=\cos B$

$B=90^0-A \Rightarrow A+B=90^0$

$\sin A.\sin B=\frac{1}{4}$

$\Rightarrow \cos B\sin B=\frac{1}{4}$ $\quad\quad \Rightarrow B=15^0 \quad A=75^0$

$\cos(A-B)=\frac{1}{2}$

4. $\sin C=\frac{AD}{b}$

$\Rightarrow \frac{abc}{b^2-c^2}=b.\sin C$

$\Rightarrow \frac{\sin A.\sin C}{\sin(B+C)\sin(B-C)}=\sin C$

$\Rightarrow \sin(B-C)=1$

$B=90^0+23^0=113^0$

5. $\frac{\sin A}{3}=\frac{\sin B}{3}=\frac{\sin C}{4}$

$\therefore a=3k, b=3k, c=4k$

$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{1}{9}$

6. $3\sin A=6\sin B=2\sqrt{3}\sin C$

$\Rightarrow \frac{\sin A}{2}=\frac{\sin B}{1}=\frac{\sin C}{\sqrt{3}}$

$\Rightarrow A=90^0, B=30^0, C=60^0$

7. $\cos(A+B)=\frac{31}{32}$

$\Rightarrow -\cos C=\frac{31}{32}$

$\Rightarrow \frac{a^2+b^2-c^2}{2ab}=\frac{31}{32}$

$\Rightarrow c=\frac{\sqrt{319}}{2}$

8. $\frac{2s2(s-a)}{bc}=\lambda$

$\Rightarrow \cos^2\left(\frac{A}{2}\right)=\frac{\lambda}{4}$

$0<\frac{\lambda}{4}<1$

9. $\Delta=(a-b+c)(a+b-c)$

$\Rightarrow \Delta=2(s-b)2(s-c)$

$$\Rightarrow \frac{(s-b)(s-c)}{\Delta} = \frac{1}{4}$$

$$Tan\left(\frac{A}{2}\right) = \frac{1}{4}$$

10. $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{A}{2} = 7+9:3+9:3+7$

$=8:6:5$

11. Verification method Put $x = y = z$ Then the triangle is equilateral

$$\Delta = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}\times\left(\frac{4}{5}\right)^2$$

$$= \frac{4\sqrt{3}}{25}$$

12. $2r_1 = a+b+c = 2s$

$\Rightarrow r_1 = s$

$\Rightarrow s\tan\left(\frac{A}{2}\right) = s$

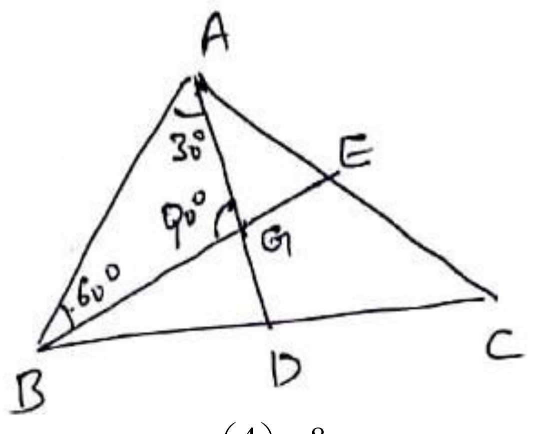
$A = 90^0$

13. $r_2 + r_3 = 4R\cos^2\left(\frac{A}{2}\right)$

$\therefore \cos^{-1}\left(\frac{R}{4R\cos^2 A/2}\right) = 60^0$

14. $a:b:c = 2+3:1+3:1+2 = 5:4:3$ $\therefore a:b = 5:4$

15.



$$AD = 4 \Rightarrow AG = 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

$$\frac{\sin 30^0}{BG} = \frac{\sin 60^0}{8/3} \Rightarrow BG = \frac{8}{3\sqrt{3}}$$

Area $\Delta ABG = \frac{1}{2}\times\frac{8}{3\sqrt{3}}.\frac{8}{3}.\sin 90^0$

Area $\Delta ABC = 3\times\frac{1}{2}\times\frac{8}{3\sqrt{3}}\times\frac{8}{3} = \frac{32}{3\sqrt{3}}$

16. $\frac{a}{\sin A} = \frac{b}{\sin B}$

$\therefore \sin B = \dfrac{b.\sin A}{a} = 8 \times \dfrac{5}{13} \times \dfrac{1}{3} = \dfrac{40}{39} > 1$

17. $TanA = TanB = TanC$

$\Delta = \dfrac{\sqrt{3}}{4}(2)^2 = \sqrt{3}$

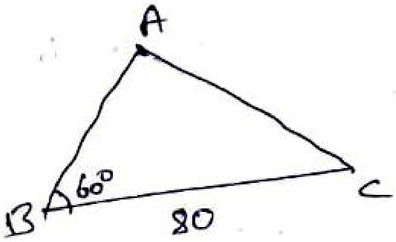
18. $r + r_1 + r_2 - r_3 = 4R\cos C = 0$

$\Rightarrow C = 90^0$

19. $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} \Rightarrow r = 2$

$c = \sqrt{(r_3 - r)(r_1 + r_2)} = 13$

20.



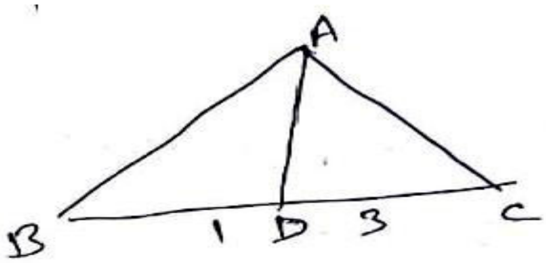
$a = 80, b + c = 90, B = 60^0$

$b^2 = a^2 + c^2 - 2ac\cos B$

$\therefore b = 73$

$\therefore c = 17$

21.



By sine rule

$\dfrac{\sin\angle BAD}{1K} = \dfrac{\sin B}{AD} ----(1)$

$\dfrac{\sin\angle CAD}{3K} = \dfrac{\sin C}{AD} --(2)$

$\therefore (1) \& (2) \Rightarrow \dfrac{\sin\angle BAD}{\sin\angle CAD} = \dfrac{1}{\sqrt{6}}$

22. $a^2 = b^2 + c^2 - 2bc\cos A$

$16 = 9 + c^2 - 2(3)c\left(\dfrac{1}{2}\right)$

$\Rightarrow c^2 - 3c - 7 = 0$

23. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos C = \dfrac{b}{a/2} = \dfrac{2b}{a} \Rightarrow \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{2b}{a} \Rightarrow 3b^2 = a^2 - c^2$

$\cos A\cos C = \dfrac{2\left(c^2 - a^2\right)}{3ac}$

24. $\dfrac{e+a}{b} = \dfrac{\cos\left(\dfrac{C-A}{2}\right)}{\sin\left(\dfrac{B}{2}\right)}$

$= 2\cos\left(\dfrac{C-A}{2}\right)\left(\because B = 60^0\right)$

25. $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$

$r_1 = s$

$\Rightarrow s\tan\dfrac{A}{2} = S$

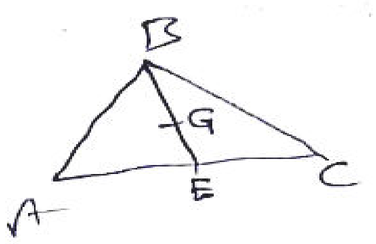
$\Rightarrow A = 90^0$

26. $AD^2 = \dfrac{1}{4}\left(2b^2 + 2c^2 - a^2\right)$

$BE^2 = \dfrac{1}{4}\left(2a^2 + 2c^2 - b^2\right)$

$CF^2 = \dfrac{1}{4}\left(2a^2 + 2b^2 - c^2\right)$

27.



$BE = \dfrac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$

$= \dfrac{1}{2}\sqrt{98 + 162 - 64} = 7$

$\therefore BG = \dfrac{2}{3}(7) = \dfrac{14}{3}$

28. $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}} = \dfrac{3}{4}$

$\therefore CF = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$

$= \dfrac{10}{3}$

29. Put $\theta = 0$

$b\cos C + c\cos B = a = a\cos\theta$

30. $a + b + c = 6\dfrac{(\sin A + \sin B + \sin C)}{3}$

$\Rightarrow R = 1$

$\therefore a = 2R\sin A$

$\sin A = \frac{1}{2}$

31. $\sin A = 2\sin B$

$\Rightarrow \frac{\sin A}{\sin B} = \frac{2}{1}$

$Tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right) = \frac{3}{1}$

$A + B = 120^0, C = 60^0$

32. $2s\left[\frac{(s-b)(s-c)}{\Delta} + \frac{(s-a)(s-c)}{\Delta}\right]$

$= \frac{2s.(s-c)}{\Delta}[s-b+s-a]$

$= 2c.\cot\frac{C}{2}$

33. $tan\left(\frac{B-C}{2}\right) + \cot\left(\frac{B-C}{2}\right)$

$= 2\cos ec2\left(\frac{B-C}{2}\right)$

34. $B + C = 120^0$

$B - C = 60^0$

$\Rightarrow B = 90^0, C = 30^0$

$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$

$\therefore \frac{b-c}{b+c} = \frac{1}{3}$

35. $c = 2R\sin C$

$\Rightarrow R = \frac{C}{2}$

$r = (s-c)\tan\frac{C}{2} = s - c$

$R + r = \frac{c}{2} + s - c$

$= \frac{c+2(s-c)}{2} = \frac{c+a+b-c}{2} = \frac{a+b}{2}$

36. $A = B = C = 60^0$

$4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : R : 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$= 4.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}:1:4.\frac{1}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} = \frac{1}{2}:1:\frac{3}{2} = 1:2:3$

37. Put $x = 2$

$a = 7, b = 5, c = 3$

$\cos A = \frac{25+9-49}{2\times5\times3} = \frac{-15}{30} = \frac{-1}{2}$

$\therefore a = 120^0$

38. $\Delta\sqrt{s(s-a)(s-b)(s-c)}$ $\quad s^1 = \frac{2a+2b+2c}{2}$

$\Delta^1 = \sqrt{s^1(s^1-2a)(s^1-2b)(s^1-2c)}$ $\quad a+b+c=2s$

$= \sqrt{2s.2(s-a)2(s-b)2(s-c)}$

$\Delta^1 = 4\Delta$

39. $s-a=11k, s-b-12k, s-c=13k$

$s=36k$

$tan^2\left(\frac{A}{2}\right) = \frac{(s-b)(s-c)}{s(s-a)} = \frac{(12k).(13k)}{(36k).11k} = \frac{13}{33}$

40. $\sqrt{3}r+a=s$

$\Rightarrow \sqrt{3}\frac{\Delta}{s}+a=s$

$\Rightarrow \sqrt{3} = \frac{s(s-a)}{\Delta} = \cot\left(\frac{A}{2}\right)$

$\therefore \frac{A}{2} = 30^0 \Rightarrow A = 60^0,\ B=60^0, C=60^0$

41. $\cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + \frac{3}{4} = \frac{7}{4}$

42. $\cos A = \frac{r}{R}$

$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

$\Rightarrow \cos B + \cos C = 1$

43. $R=1 \Rightarrow a = 2R\sin A = 2\sin A$

Max. value is $a=2$

44. For $A=B=45^0, C=90^0$ the given equation is satisfied

$a:b:c = \sin A : \sin B : \sin C = \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 1 = 1:1:\sqrt{2}$

$\Rightarrow k = \sqrt{2} \Rightarrow \frac{k^4+10k^2}{6} = 4$

45. $(a-2b)^2 + (a-c)^2 = 0 \Rightarrow a=2b, a=c$

$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{7b^2}{8b^2} = \frac{7}{8}\ 8\cos B = 7$

46. $(b^2+c^2-a^2)^2 = 3b^2c^2 \Rightarrow b^2+c^2-a^2 = \sqrt{3}bc$

$\Rightarrow \frac{b^2+c^2-a^2}{2bc} = \frac{\sqrt{3}}{2} \Rightarrow \cos A = \frac{\sqrt{3}}{2} \Rightarrow 2\sqrt{3}\cos A = 3$

47. $a^2+b^2-c^2 = 100c^2 \Rightarrow \frac{a^2+b^2-c^2}{2ab} = \frac{100c^2}{2ab}$

$\Rightarrow \cos C = \frac{50\ 2R\sin C 2R\sin C}{2R\sin A 2\ R\sin B}$

$$\Rightarrow \cot C = 50\frac{\sin(A+B)}{\sin A\sin B} = 50(\cot A + \cot B)$$

48. $\frac{r}{R} = \frac{1}{8} \Rightarrow 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{8}$

$$\Rightarrow \left(\cos\left(\frac{A}{2}-\frac{B}{2}\right) - \cos\left(\frac{A}{2}+\frac{B}{2}\right)\right)\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left(\frac{1}{2} - \sin\frac{C}{2}\right)\sin\frac{C}{2} = \frac{1}{16} \Rightarrow \sin\frac{C}{2} = \frac{1}{4}$$

$$8\cos C = 8\left(1 - 2\sin^2\frac{C}{2}\right) = 7$$

49. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \Rightarrow r = 2$

$$r_1 + r_2 + r_3 - r = 4R \Rightarrow R = \frac{13}{2}$$

50. $r + r_2 + r_3 - r_1 = 4R\cos A = 0 \Rightarrow A = 90^0$

$$a^2 = b^2 + c^2 = 169 \Rightarrow a = 13$$

51. $\sin 30^0 = \frac{6}{x} \Rightarrow x = 12$

52. $h = \frac{d}{\cot\alpha - \cot\beta} = \frac{60}{\sqrt{3}-1} = 30\left(\sqrt{3}+1\right)$

53. $\tan\theta = \frac{x}{d}, \ \tan 2\theta = \frac{3x}{d}$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{3x}{d} \Rightarrow \frac{x}{d} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

54. $\cot\alpha = \frac{a+b+x}{h} \Rightarrow a + b + x = h\cot\alpha$

$$\cot 2\alpha = \frac{b+x}{h} \Rightarrow b + x = h\cot 2\alpha$$

$$\cot 3\alpha = \frac{x}{h} \Rightarrow x = h\cot 3\alpha$$

$$\Rightarrow a = h(\cot\alpha - \cot 2\alpha), \ b = h(\cot 2\alpha - \cot 3\alpha)$$

$$\frac{a}{b} = \frac{\sin 3\alpha}{\sin\alpha} = 1 + 2\cot 2\alpha$$

55. $\cot\theta = \frac{AP}{2h}, \ \cot\left(90^0 - \theta\right) = \frac{PC}{h}$

$$\Rightarrow \cot\theta\tan\theta = \frac{60}{2h}.\frac{60}{h}$$

$$\Rightarrow h^2 = 180 \Rightarrow h = 30\sqrt{2} \Rightarrow 2h = 60\sqrt{2}$$

56. $AD = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}}{2}a$

$\tan\alpha = \frac{h}{AD} \Rightarrow AD = h\cot\alpha$

$\frac{\sqrt{3}}{2}a = h\cot\alpha \Rightarrow a = \frac{2h}{\sqrt{3}}\cot\alpha$

57. $\cot\alpha = \frac{R}{h_1}, \cot\beta = \frac{R}{h_2}, \cot\gamma = \frac{R}{h_3}$

$\cot\alpha, \cot\beta, \cot\gamma$ are in A.P

$\Rightarrow \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}$ are in A.P

58. $\sin 30^0 = \frac{a}{CP} \Rightarrow CP = 2a$

59. Height of the cloud is $h\left(\frac{\cot\theta + \cot 45^0}{\cot\theta - \cot 45^0}\right) = h\tan\left(\frac{\pi}{4} - \theta\right)$

60. $\tan\alpha = \frac{x/2}{nx} = \frac{1}{2n}$, $\tan(\alpha+\beta) = \frac{x}{nx} = \frac{1}{n}$

$\tan\beta = \tan(\alpha+\beta-\alpha) = \frac{\tan(\alpha+\beta) - \tan\alpha}{1 + \tan(\alpha+\beta)\tan\alpha} = \frac{n}{2n^2+1}$

COMPLEX NUMBERS

1. True when $n = 4, 8, 12....$
   When n is divided with 4, we get 0 as remainder
2. We have $iz^2 = (\bar{z})^2 + z - (1)$
   Taking conjugate to $(1) \to -i(\bar{z})^2 = z^2 + \bar{z} - (2)$
   Eliminating z^2 from $(1),(2) \to iz = \bar{z}$
   Put $z = x + iy$ in above equation we get $x = -y \Rightarrow z = x(1-i)$
   Hence $z = \frac{1-i}{2}$
   $\Rightarrow |z| = \frac{1}{\sqrt{2}}$
3. Assume $\frac{z-1}{z+1} = kl$
   Apply componendo and dividendo $\to z = -\frac{1+ik}{1-ik} \Rightarrow |z| = 1$
4. True when $b = 0, d = 0$
5. z_1, z_2, z_3 are collinear because of $z_1 + z_3 = 2z_2$
6. Given relation is $PA + PB$ and $(PA+PB)_{\min i}$ =AB=5
7. Take $z = \cos\theta + i\sin\theta$ $(\textit{such that } |z| = 1)$ *and simplify*
8. $|z_1 z_2| = 1 \Rightarrow |\bar{z}_1 z_2| = 1$

arg z_1 -arg $z_2 = \frac{\pi}{2} \Rightarrow -Arg\ \overline{z}_1 - Arg\ z_2 = \frac{\pi}{2}$

$\Rightarrow Arg\ \overline{z}_1 + Arg\ z_2 = -\frac{\pi}{2}$

$\Rightarrow Arg\left(\overline{z}_1 z_2\right) = \frac{-\pi}{2}$

Now mod-arg form $\Rightarrow \overline{z}_1 z_2 = r\ cis\theta$

$\begin{pmatrix} where\ r = \left|\overline{z}_1 z_2\right| = 1 \\ \theta = Arg\left(\overline{z}_1 z_2\right) = \frac{-\pi}{2} \end{pmatrix}$

$\Rightarrow \overline{z}_1 \overline{z}_2 = (1) cis\left(\frac{-\pi}{2}\right) \Rightarrow \overline{z}_1 z_2 = -i$

9. Assume $z = (x_1, y_1)\ and\ \omega = (x_2, y_2)$

Now Re $z = |z-2|$, Re $\omega = |\omega - 2|\ gives$

$4x_1 = 4 + y_1^2, 4x_2 = 4 + y_2^2$

$\Rightarrow 4x_1 - 4x_2 \Rightarrow y_1^2 - y_2^2 \Rightarrow \frac{y_1 - y_2}{x_1 - x_2} = \frac{4}{y_1 + y_2} - (1)$

We have Arg $(z - \omega) = \frac{\pi}{3} \Rightarrow Tan^{-1}\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \frac{\pi}{3} \Rightarrow \frac{y_1 - y_2}{x_1 - x_2} = Tan\frac{\pi}{3}$

$\Rightarrow \frac{y}{y_1 + y_2} = \sqrt{3}$

$\therefore (1)\ \ y_1 + y_2 = \frac{4}{\sqrt{3}} \Rightarrow I_M(z + w) = \frac{1}{\sqrt{3}}$

10. $\overline{z}\ z^3 + z\ \overline{z}^3 = 350 \rightarrow z\ \overline{z}\left(z^2 + \overline{z}^2\right) = 350$

$\Rightarrow \left(x^2 + y^2\right)\left(x^2 - y^2\right) = 25 \times 7$

$\Rightarrow x^2 + y^2 = 25\ and\ x^2 - y^2 = 7 \Rightarrow x = \pm 4, y = \pm 3$

Vertices of ABCD are given by $(\pm 4, \pm 3)$

$\Rightarrow A(4,3) B(-4,3), C(-4,-3), D(4,-3)$

Now Area is $AB \times BC = 8 \times 6 = 48$

11. We have $ab = 1 + \sqrt{1 - b^2} \Rightarrow a = \frac{1}{b} + \sqrt{\frac{1}{b^2} - 1}$

$\Rightarrow a = \cos\alpha + i\sin\alpha - (1)$

$\left(where\ \frac{1}{b} = \cos\alpha\ (ie) b\cos\alpha = 1\right)$

Now $\frac{b}{2a}(1 + az)(1 + az) = \frac{b}{2}\left(\frac{1}{a} + z\right)(1 + az)$

$= \frac{b}{2}\left(a + \frac{1}{a} + z + \frac{1}{z}\right) = \frac{b}{2}\left(e^{i\alpha} + e^{-i\alpha} + e^{i\theta} + e^{-i\theta}\right) = \frac{b}{2}(2\cos\alpha + 2\cos\theta)$

$= 1 + b\cos\theta$

12. $|3 + i(z - 1)| = ||i(z - 1 - 3i)| = |z - 1 - 3i|$

$= |(z - 2i) + (-1 - i)|$

$\leq |z - 2i| + |1 + i|$

$\le \sqrt{2}+\sqrt{2}=2\sqrt{2}$

$\Rightarrow \left|3+i(z-1)\right| \le 2\sqrt{2}$

$\Rightarrow$ Maximum value of $\left|3+i(z-1)\right| is\ 2\sqrt{2}$

13. $\dfrac{1+iz}{1-iz}=\dfrac{1+i\left\{\dfrac{b+ic}{1+a}\right\}}{1=i\left\{\dfrac{b+ic}{1+a}\right\}}=\dfrac{(1+a-c)+lb}{(1+a+c)+lb}$

$\Rightarrow \dfrac{\left[(1+a-c)+ib\right]\left[(1+a+c)-ib\right]}{(1+a+c)^2+b^2}$

$=\dfrac{2a+2a^2+2ib+2iab}{2+2ac+2(a+c)}$ (After using $a^2+b^2+c^2=1$)

$=\dfrac{a(a+1)+ib(a+1)}{(a+1)(c+1)}=\dfrac{a+ib}{1+c}$

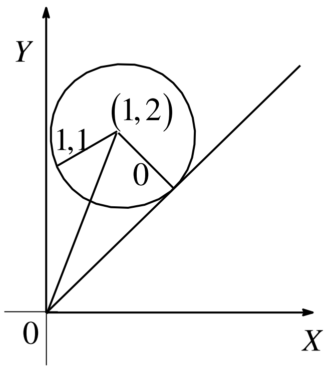
14. we have $(z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2=0$

$\Rightarrow z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

$\Rightarrow A(z_1),B(z_2),C(z_3)$ are vertices of equilateral triangle

$\Rightarrow AB=BC=CA \rightarrow |z_1-z_2|=|z_2-z_3|=|z_3-z_1|$

15.



$\tan\theta=\tan\left((\theta+\alpha)-\alpha\right)=\dfrac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha}\quad \dfrac{\frac{2}{1}-\frac{1}{2}}{1+\left(\frac{2}{1}\right)\left(\frac{1}{2}\right)}=\dfrac{3}{4}$

$\Rightarrow \theta=\tan^{-1}\left(\dfrac{3}{4}\right)$

16. Observe $\dfrac{2+i}{3-i}=P\left(\dfrac{1}{2},\dfrac{1}{2}\right)$

$Also(1+i)z=(1-i)\bar{z}\Rightarrow x+y=0$

Reflection of $P\left(\dfrac{1}{2},\dfrac{1}{2}\right)$ write the line $x+y=0$

Is $Q\left(\dfrac{-1}{2},\dfrac{-1}{2}\right)=\dfrac{2+i}{-3+i}$

17. let $z=r(\cos\theta+\sin\theta)$

Now $\left|5z+\dfrac{1}{z}\right|=3\Rightarrow\left|5r\cos\theta+5ir\sin\theta+\dfrac{\cos\theta}{r}-\dfrac{i\sin\theta}{r}\right|=3$

$$\Rightarrow\left(5r\cos\theta+\frac{\cos\theta}{r}\right)^2+\left(5r\sin\theta-\frac{\sin\theta}{r}\right)^2=9$$

$$\Rightarrow 25r^2+\frac{1}{r^2}+10\left(\cos^2\theta-\sin^2\theta\right)=9$$

$$\Rightarrow\cos 2\theta=\frac{9-\left(25r^2+\frac{1}{r^2}\right)}{10}-(i)$$

$$But\ 25r^2+\frac{1}{r^2}\geq 10$$

$$(\therefore AM\geq GM)$$

$$\Rightarrow 9-\left(24r^2+\frac{1}{r^2}\right)\leq -1$$

$$\Rightarrow\frac{9-\left(24r^2+\frac{1}{r^2}\right)}{10}\leq\frac{-1}{10}\Rightarrow\cos 2\theta\leq\frac{-1}{10}$$

$\therefore Max$ Value of $\cos 2\theta\ is\ \frac{-1}{10}$

18. $5=\left|z-\frac{6}{z}\right|\geq|z|-\frac{6}{|z|}$

$$\Rightarrow 5\geq|z|-\frac{6}{|z|}\Rightarrow|z|^2-5|z|-6\leq 0$$

$$\Rightarrow|z|\in[-1,6]$$

$\therefore$ Max value of $|z|is\ 6$

19. Data $\Rightarrow|a|=|b|=|c|\Rightarrow a\bar{a}=b\bar{b}=c\bar{c}=1$

$$\Rightarrow\bar{a}=\frac{1}{a},\bar{b}=\frac{1}{b},\bar{c}=\frac{1}{c}$$

$$\Rightarrow\left|\bar{a}+\bar{b}+\bar{c}\right|=\left|\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right|$$

$$\Rightarrow|a+b+c|=\left|\frac{bc+ca+ab}{abc}\right|$$

$$\Rightarrow|abc|=\frac{|bc+ca+ab|}{|abc|}$$

$$\Rightarrow\left(|a||b||c|\right)^2=|ab+bc+ca|$$

$$\Rightarrow(1\times1\times1)^2=|ab+bc+ca|$$

$$\Rightarrow|ab+bc+ca|=1$$

20. $-\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\left(2\sin\frac{x}{2}\right)+\tan x=0$

$$\Rightarrow(\tan x-1)(1-\cos x)=0\Rightarrow x=0,2\pi,\frac{\pi}{4},\frac{5\pi}{4}$$

$$\Rightarrow sum\ is\ \frac{7\pi}{2}$$

21. Centres are $c_1(-1),c_2(-c)$ and radii are $r_1=\sqrt{a\bar{a}-b}$, $r_2=\sqrt{c\bar{c}-d}$

We have circles cut othogenerally

$\Rightarrow c_1c_2^2 = r_1^2 + r_2^2$

$\Rightarrow |a-c|^2 = (a\bar{a}-b)+(c\bar{c}-d)$

$\Rightarrow (a-c)(\overline{a-c}) = |a|^2 - b + |c|^2 - d$

$\Rightarrow |a|^2 + |c|^2 - a\bar{c} - \bar{a}c = |a|^2 + |c|^2 - b - d$

$\Rightarrow a\bar{c} + \bar{a}c = b + d$

$\Rightarrow 2\,\mathrm{Re}(a\bar{c}) = b + d \ (\therefore z + \bar{z} = 2\,\mathrm{Re}(z))$

$\Rightarrow \mathrm{Re}(a\bar{c}) = \dfrac{b+d}{2}$

22. Consider $z = rcis\theta \Rightarrow \bar{z} = rcis(-\theta)$

$\Rightarrow \dfrac{z-\bar{z}}{2021} = r\dfrac{(zi\sin\theta)}{2021}$

$\Rightarrow \dfrac{z-\bar{z}}{2021} = -\left(\dfrac{2r\sin\theta}{2021}\right)i$

$\Rightarrow \dfrac{\bar{z}-z}{2021} = ki(where\ k\ is + ve)$

$\Rightarrow Arg\left(\dfrac{\bar{z}-z}{2021}\right) = \dfrac{\pi}{2}$

23. $\cos ec^{-1}\left(\dfrac{z}{1+x}\right) = \cos ec^{-1}\left(\dfrac{(x+iy)(1-i)}{1^2-i^2}\right)$

$= \cos ec^{-1}\left(\dfrac{x+y}{2} + i\left(\dfrac{y-x}{2}\right)\right) - (1)$

Above (1) exists iff $\dfrac{y-x}{2} = 0$ and $\dfrac{x+y}{2} \le -1 (or) \dfrac{x+y}{2} \ge 1$

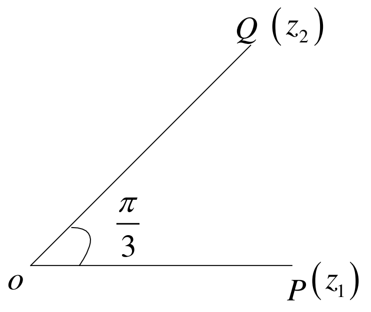
$(ie) x = y\ and\ 2x \le -2 (or) 2x \ge 2$

$(ie) x = y\ and\ x \le -2 (or) x \ge 1$

(ie) Both x, y have same sign

(ie) z Lies in 1^{st} or 3^{rd} quadrants

24.



We have $(z_2 - 0) = (z_1 - 0)e^{i\frac{\pi}{3}}$

$\Rightarrow z_2^3 = z_1^3, e^{i\pi}$

$\Rightarrow z_2^3 = -z_1^3 \Rightarrow z_1^3 + z_2^3 = 0$

25. Data $\Rightarrow |z_1 - 0| = |z_2 - 0| = |z_3 - 0| = 1$

$\dfrac{z_1 + z_2 + z_3}{3} = 0$

$\Rightarrow \Delta ABC$ is equilateral $\Rightarrow A = B = C = 60^0$

Now $|z_1-0|=|z_2-0|=|z_3-0|=1 \Rightarrow R=1$

We have $\Delta = 2R^2 \sin A \sin B \sin C$

$= 2(1)^2 \sin 60^0 \sin 60^0 \sin 60^0$

$= \dfrac{3\sqrt{3}}{4}$ *Square units*

26. $(z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3) = (\sqrt{2}+i)(\sqrt{2}-i)$

$\Rightarrow |z_1|^2+|z_2|^2+|z_3|^2+(z_1\bar{z}_2+z_2\bar{z}_3+z_3\bar{z}_1)$

$(z_1\bar{z}_2+\bar{z}_2 z_3+\bar{z}_3 z_1) = 2+1$

$\Rightarrow 1+1+1+\alpha+\bar{\alpha}=3$

(Where $\alpha = z_1\bar{z}_2+z_2\bar{z}_3+z_3\bar{z}_1$)

$\Rightarrow \alpha+\bar{\alpha}=0 \Rightarrow 2\operatorname{Re}\alpha = 0 \Rightarrow \operatorname{Re}\alpha=0$

$\Rightarrow \alpha$ is purely imaginary

$\Rightarrow z_1\bar{z}_2+z_2\bar{z}_3+z_3\bar{z}_1$ is purely imaginary

27. $|z-2i|=2$ Is a circle with centre $2i$

$z(1-i)-\bar{z}(1+i)=4i$ Is a line and the line passes through centre of the circle, hence they intersect in 2 points

28. *Let* $z=x+iy$

Now $E = z\bar{z}+(z-3)(\bar{z}-3)+(z-6i)(z+6i)$

$=3z\bar{z}-3(z+\bar{z})+6(z-\bar{z})i+45$

$3(x^2+y^2-2x-4y+15)$

$=3((x-1)^2+(y-2)^2+10)$

Thus $E=3((x-1)^2+(y-2)^2+10)$

Hence $E_{\min}=30,$ Occurs when $x=1, y=2$

$z=1+2i$

29. *Let* $z=x+iy$

Now $E = z\bar{z}+(z-3)(\bar{z}-3)+(z-6i)(z+6i)$

$=3z\bar{z}-3(z+\bar{z})+6(z-\bar{z})i+45$

$3(x^2+y^2-2x-4y+15)$

$=3((x-1)^2+(y-2)^2+10)$

Thus $E=3((x-1)^2+(y-2)^2+10)$

Hence $E_{\min}=30,$ Occurs when $x=1, y=2$

$\therefore$ *Minimum* Value of E is 30 and this occurs at $(x,y)=(1,2)=1+2i$

30. $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2) = |z_1|^2+|z_2|^2$

$\Rightarrow 2\cos\theta(\theta_1-\theta_2)=0 \Rightarrow \theta_1-\theta_2=\dfrac{\pi}{2}$

$\Rightarrow Amp\ z_1 - Amp\ z_2 = \dfrac{\pi}{2}$

$\Rightarrow Amp\left(\frac{z_1}{z_2}\right)=\frac{\pi}{2}\Rightarrow\frac{z_1}{z_2}$ is purely imaginary

31. Let $z=\lambda cis\theta$, then given equation is $\lambda^2+\frac{4}{\lambda^2}-2.2\cos 2\theta-16=0$

$\Rightarrow\lambda^2-4\lambda^2(c\text{os}2\theta+4)+4=0$

Using quadratic formula $\lambda^2=2(\cos 2\theta+4)$

$+2\sqrt{(\cos 2\theta+4)^2-1}$

$\therefore$ Maximum value $\lambda^2=10+2\sqrt{24}=\left(2+\sqrt{6}\right)^2$

$\Rightarrow$ Max value of λ is $2+\sqrt{6}$

32. $z,-z,(1-z)$ forms vertices of an equilateral triangle

$\Rightarrow z^2+(-z)^2+(1-z)^2=(z)(-z)(1-z)+(1-z)z$

$\Rightarrow 3z^2-2z+1=-z^2\Rightarrow 4z^2-2z+1=0$

$\Rightarrow z=\frac{2\pm\sqrt{4-16}}{8}\Rightarrow z=\frac{2+2i\sqrt{3}}{4}$

$\Rightarrow \text{Re}(z)=\frac{2}{8}=\frac{1}{4}$

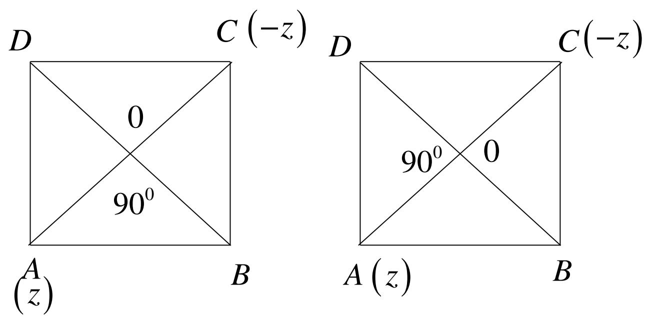
33. Let $z=x+iy\Rightarrow\overline{z}=x-iy$

So the vertices will be $(x,y),(x,-y),(-x,-y),(-x,y)$

So sides of square will be $2x,2y$: which gives $2x=2y=4(ie)x=y=2$

Now $z=x+iy=2+2i\Rightarrow|z|=\sqrt{4+4}\Rightarrow|z|=2\sqrt{2}$

34.



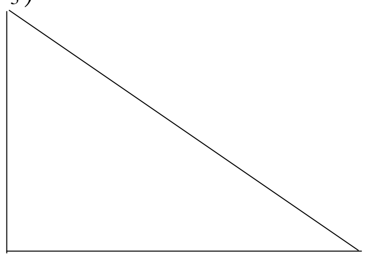
A,B,C,D are in clockwise order

From the above diagrams $\Rightarrow A,B,C$ are $z_1-z_1\ ze$

$\Rightarrow A,B,C\ are\ z_1,-z_1\pm iz$

$\Rightarrow$ Centroid of ΔABC is $\frac{\pm iz}{3}$

35.



AC is obtained by rotating AB bout A through an angle $\frac{\pi}{2}$,thus

$$\frac{z_1-z_3}{|AC|}=\frac{z_1-z_2}{|AB|}e^{I\frac{\pi}{2}}$$

$$\Rightarrow (z_1-z_2)=i(z_1-z_2)$$

$$(by\ data\ |AB|=|AC|)$$

$$\Rightarrow z_3=(1-i)z_1+iz_2$$

$$(Like\ z_3=\alpha z_1+\beta z_2)$$

$$\Rightarrow \alpha=1-i,\beta=i\Rightarrow|\alpha|+|\beta|=\sqrt{2}+1$$

36. Observe $BA=\begin{bmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z} \end{bmatrix}\begin{bmatrix} \overline{z}_1 & -z_2 \\ \overline{z}_2 & z_1 \end{bmatrix}=\begin{bmatrix} |z_1|^2+|z_2|^2 & 0 \\ 0 & |z_1|^2+|z_2|^2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$$Now\,(BA)^{-1}=\frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1}B^{-1}=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

37. Put $x=x+iy\ \left(|z|=\sqrt{x^2+y^2},\theta=\tan^{-1}\frac{y}{x}\right), we\ get$

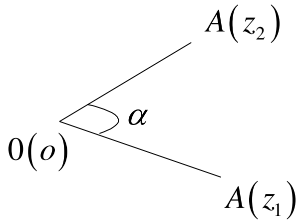
$$\sqrt{(x-2)^2+(y-1)^2}=\sqrt{x^2+y^2}\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{2}\tan\theta\right)$$

$$=\frac{\sqrt{x^2+y^2}}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+y^2}}-\frac{y}{\sqrt{x^2+y^2}}\right)=\frac{x-y}{\sqrt{2}}$$

$$\Rightarrow\sqrt{(x-2)^2+(y-1)^2}=(i)\left(\frac{x-y}{\sqrt{2}}\right)\Rightarrow SP=ePM\ (e=1)$$

$\therefore$ *Locus* of P is parabola with focus $(2,1)$ and directrix is $x-y=0$

38.



Data $\Rightarrow z_1 + z_2 = -p, z_1 z_2 = q, z_2 = z_1 e^{i\alpha}$

Now $\frac{z_2}{z_1} = e^{i\alpha} \Rightarrow \frac{z_2}{z_1} + 1 + (\cos\alpha + i\sin\alpha)$

$$\Rightarrow \frac{z_1 + z_2}{z_1} = 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

$$\Rightarrow \left(\frac{z_1 + z_2}{z_1}\right)^2 = 4\cos^2\left(\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)^2$$

$$\Rightarrow (z_1 + z_2)^2 = 4z_1^2\cos^2\left(\frac{\alpha}{2}\right)e^{i\alpha}$$

$$\Rightarrow (z_1 + z_2)^2 = \left(4z_1^2\cos^2\left(\frac{\alpha}{2}\right)\right)\left(\frac{z_2}{z_1}\right)\left(\therefore e^{i\alpha} = \frac{z_2}{z_1}\right)$$

$$\Rightarrow (z_1 + z_2)^2 = 4z_1 z_2\cos^2\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow (-p)^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$$

39. Assume $z_1 = (\cos\alpha + i\sin\alpha), z_2 = 2(\cos\beta + i\sin\beta)$ and $z_3 = 3(\cos\gamma + i\sin\gamma)$

Then we get $z_1 + z_2 + z_3 = 0$

Now $z_1 + z_2 + z_3 = 0 \Rightarrow z_1^3 + z_2^3 + z_3^3 = 3z_1 z_2 z_3$

$\Rightarrow cis\ 3\alpha + 8\ cis 3\beta + 27 cis\ 3r = 18 cis(\alpha + \beta + \gamma)$

Equating real parts $\Rightarrow \sin 3\alpha + 8\sin 3\beta + 27\sin 3r = 18\sin(\alpha + \beta + \gamma)$

40. We have $z_n = e^{\frac{i\pi}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)}$

41. Now $z_1 z_2 z_3 ......z_n = e^{\frac{i\pi}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + ... + \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)}$

Replace ℓ by 2021-ℓ in (1), we get

$$S = \sum_{r=1}^{2020}(2021)(2)(\alpha_1)$$

$$\Rightarrow 2S = 2(202)\sum_{r=1}^{2020}\alpha_r$$

$$\Rightarrow S = 2021(\alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{2020})$$

$$\Rightarrow S = 2021(1 + \alpha_1 + \alpha_2 + ... + \alpha_{2020} - 1)$$

$$\Rightarrow S = 2021(0 - 1) \Rightarrow S = -2021$$

42. We have $\alpha^7 = 1 \Rightarrow 1 + \alpha + \alpha^2 + ... + \alpha^6 = 0$

Now on simplification ,we get $(a + b) = -1\ ab = 2$

$\therefore |(a + b) + abi| = |-1 + 2i| = \sqrt{5}$

43. $i^4, i^5 .... are\, 1,1.....\left(As\ i^4 = 1\right)$

$\omega^{3!}, \omega^{4!}, \omega^{5!} ....are\ 1,1,1.......\left(As\ \omega^3 = 1\right)$

Now $\sum_{k=1}^{100} i^{k!} = i + i^2 + i^6 + 1 + 1 + ... + 1 = 95 + i$

$\sum_{k=1}^{100} \omega^{k!} = \omega + \omega^2 + 1 + 1 + ... + 1 = 97$

$\therefore \sum_{k=1}^{100}\left(l^{k!} + \omega^{k!}\right) = \sum_{k=1}^{100} i^{k!} + \sum_{k=1}^{100} \omega^{k!}$

$= (95 + i) + 97 = 192 + i$

44. We have $T_n = (n+1)(n\omega+1)(n\omega^2+1)$

$T_n = (n+1)(n^2 - n + 1)$

$T_n = n^3 + 1$

Now $S_n \sum T_n = \sum\left(n^3+1\right)$

$S_n = \dfrac{n^3(n+1)^2}{4} + n$

Put $n = 24 \Rightarrow S_{24} = \dfrac{(24)^2(25)^2}{4} + n$

$= 24\{(6\times 25\times 25) + 1\}$

$= 24(........1)$

$\rightarrow$Units place must be 4

45. $\left|a + b + c\omega^2\right| = \left|a + b\left(\dfrac{-1}{2} + \dfrac{i\sqrt{3}}{2}\right) + c\left(\dfrac{-1}{2} - \dfrac{i\sqrt{3}}{2}\right)\right|$

$\dfrac{1}{2} = \left|(2a - b - c) + i\left(b\sqrt{3} - c\sqrt{3}\right)\right|$

$\dfrac{1}{2} = \sqrt{(2a-b-c)^2 + \left(b\sqrt{3} - c\sqrt{3}\right)^2}$

$= \dfrac{1}{\sqrt{2}}\sqrt{(a-b)^2 + (b-c)^2 + (c-a)^2} \quad -(1)$

As per data, (1) attains minimum value when $a - b = 0, b - c = 1, c - a = 1;$

Thus $\left|a + b\omega + c\omega^2\right| = \dfrac{1}{\sqrt{2}}\sqrt{o^2 + 1^2 + 1^2}$

$= \dfrac{1}{\sqrt{2}}\left(\sqrt{2}\right) = 1$

46. We have $z^{12} + 9z^6 - 400 = 0$

$\Rightarrow \left(z^6 + 25\right)\left(z^6 - 16\right) = 0$

$\Rightarrow z = (-25)^{\frac{1}{6}}\,(or)\, z = 16^{\frac{1}{6}}$

$\Rightarrow z = (25)^{\frac{1}{6}}\,(or)\, z = 16^{\frac{1}{6}}$

$\Rightarrow z = (25)^{\frac{1}{6}}(-1)^{\frac{1}{6}}\,(or)\, z = (16)^{\frac{1}{6}}(1)^{\frac{1}{6}}$

$\Rightarrow z = (25)^{\frac{1}{6}}\left(cis(2k_1+1)\frac{\pi}{6}\right)(k_1 = 0,1,2,....5)$

$z = (16)^{\frac{1}{6}}\left(cis\frac{2k_2\pi}{6}\right)\ (k_2 = 0,1,2,...5)$

From above, least +ve argument of z occurs at $k_1 = 0$

$Now\ \theta =$ (Least positive argument at $k_1 = 0$) $= \frac{\pi}{6}$

Thus $4\cos^2\theta = 4\cos^2\frac{\pi}{6} = 4\left(\frac{\sqrt{3}}{2}\right)^2 = 3$

47. Given $\alpha^5 = 1 \Rightarrow 1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$

$\Rightarrow 1+\alpha+\alpha^2+\alpha^3 = -\alpha^4 --(1)$

Now $1 = \alpha+\alpha^2+\alpha^3-\frac{2}{\alpha} = -\alpha^4-\frac{2}{\alpha}(\therefore \theta)$

$= \frac{-\alpha^5}{\alpha}-\frac{2}{\alpha} = -\frac{3}{\alpha}$

$\Rightarrow \left|1+\alpha+\alpha^2+\alpha^3-\frac{2}{\alpha}\right| = \left|-\frac{3}{\alpha}\right|$

$= \frac{3}{|\alpha|} = \frac{3}{1} = 3$

$\therefore \log_{\sqrt{3}}\left|1+\alpha+\alpha^2+\alpha^3-\frac{2}{\alpha}\right| = \log_{\sqrt{3}}^{3} = 2$

48. Observe that $p+q+r = 0---(1)$

$Also\ have\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r} = \frac{1}{a+b\omega+c\omega^2}+\frac{1}{b+c\omega+a\omega^2}+\frac{1}{c+a\omega+b\omega^2}$

$= \frac{1}{a+b\omega+c\omega^2}+\frac{\omega}{a+b\omega+c\omega^2}+\frac{\omega^2}{a+b\omega+c\omega^2}$

$= \frac{1+\omega+\omega^2}{a+b\omega+c\omega^2} = 0$

$\Rightarrow \frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 0 \rightarrow pq+qr+rp = 0---(2)$

Now $(1) \Rightarrow (p+q+r)^2 = 0 \Rightarrow p^2+q^2+r^2+2(pq+qr+rp) = 0$

$\Rightarrow p^2+q^2+r^2+2(0) = 0 \rightarrow p^2+q^2+r^2 = 0$

49. Given $\alpha = cis\frac{2\pi}{11} \Rightarrow \alpha^{11} = 1$

$\Rightarrow 1+\alpha+\alpha^2+......+\alpha^{10} = 0$

$\Rightarrow \alpha+\alpha^2+\alpha^3+........+\alpha^{10} = -1----(1)$

Assume $S = \lambda+\lambda^2+\lambda^3+\lambda^4+\lambda^5\left(where\ \lambda = \lambda^6\right)$

$\Rightarrow S = \alpha^6+\alpha^{12}+\alpha^{18}+\alpha^{24}+\alpha^{30}$

$\Rightarrow S = \alpha^6+\alpha+\alpha^7+\alpha^2+\alpha^8$ ___$(2)\left(\therefore \alpha^{11} = 1\right)$

$\Rightarrow S = \overline{\alpha^6}+\overline{\alpha}+\overline{\alpha^7}+\overline{\alpha^2}+\overline{\alpha^8}$

$\Rightarrow \overline{S} = \alpha^5+\alpha^{10}+\alpha^4+\alpha^9+\alpha^3$ ____(3)

$\left(\therefore \alpha \text{ is } 11^{th} \text{ root of units} \Rightarrow \overline{\alpha^i} = \alpha^{11-i}\right)$

Adding $(2),(3) \Rightarrow S+\overline{S} = \alpha+\alpha^2+\alpha^3+......+\alpha^{10}$

$\Rightarrow 2\operatorname{Re}(S) = -1(\therefore (1))$

$\Rightarrow \operatorname{Re}(S) = -\frac{1}{2}$

$\Rightarrow \operatorname{Re}\left(\lambda+\lambda^2+\lambda^3+\lambda^4+\lambda^5\right) = \frac{-1}{2}$

50. Given $z+\frac{1}{z} = \sqrt{3} \Rightarrow z^2-\sqrt{3}z+1=0$

$\Rightarrow z = cis\left(\pm\frac{\pi}{6}\right)$

Now $z^r+\frac{1}{z^r} = cis\left(\pm\frac{r\pi}{6}\right)+cis\left(\mp\frac{r\pi}{6}\right)$

$\Rightarrow z^r+\frac{1}{z^r} = 2\cos\left(\frac{r\pi}{6}\right) \Rightarrow \left(z^r+\frac{1}{z^r}\right)^2 = 4\cos^2\left(\frac{r\pi}{6}\right)$

$\Rightarrow \left(z^r+\frac{1}{z^r}\right)^2 = 2\left(1+\cos\frac{r\pi}{3}\right)$

$\Rightarrow \sum_{r=1}^{5}\left(z^r+\frac{1}{z^r}\right)^2 = 2\sum_{r=1}^{5}\left(1+\cos\frac{r\pi}{3}\right)$

$= 2\left(5+\cos\frac{\pi}{3}+\cos\frac{2\pi}{3}+\cos\frac{3\pi}{3}+\cos\frac{4\pi}{3}+\cos\frac{5\pi}{3}\right)$

$= 2(5-1) = 8$

51. We have $\frac{1+i}{1-i} = i$ (after simplification)

Now $\left(\frac{1+i}{1-i}\right)^{\frac{m}{2}} = \left(\frac{1+i}{1-i}\right)^{\frac{n}{3}} = 1$ happens when $m=8, n=12$.

Now GCD (m,n)=GCD (8,12)=4

52. We have $u = \frac{2z+i}{z-ki} = \frac{2(x+iy)+i}{(x+iy)-ki}$

$= \frac{2x+(2y+1)i}{x+(y-k)i}$

$= \frac{2x^2+(2y+1)(y-k)}{x^2+(y-k)^2} + i\frac{x(2y+1)-2x(y-k)}{x^2+(y-k)^2}$

By data $\operatorname{Re}(u)+\operatorname{Im}(u) = 1$

$\Rightarrow \frac{2x^2+(2y+1)(y-k)}{x^2+(y-k)^2} + \frac{x(2y+1)-2x(y-k)}{x^2+(y-k)^2} = 1$

Above curve cuts y-axis, so put x = 0

$\Rightarrow \frac{(2y+1)(y-k)}{(y-k)^2} = 1 \Rightarrow (2y+1)(y-k)-(y-k)^2 = 0$

$\Rightarrow (y-k)\left[(2y-1)-(y-k)\right] = 0$

$\Rightarrow (y-k)(y+k+1)=0 \Rightarrow y=k$ or $y=-(k+1)$

Thus $P=(k,0), Q=(-k-1,0)$

Given $PQ=5 \Rightarrow (2k+1)^2=25 \Rightarrow k=2, k=-3$

$(\text{But}\, k>0)$

53. Data $\Rightarrow |z-i|=|z+2i| \Rightarrow y=\dfrac{-1}{2}$

Now $z=x+iy \Rightarrow z=x-\dfrac{i}{2}\left(\because y=\dfrac{-1}{2}\right)$ …. (1)

But $|z|=\dfrac{5}{2} \Rightarrow \sqrt{x^2+\dfrac{1}{4}}=\dfrac{5}{2} \Rightarrow x^2=6 \Rightarrow x=\pm\sqrt{6}$

Now $z=(x,y)=\left(\pm\sqrt{6},\dfrac{-1}{2}\right) \Rightarrow z+3i=\left(\pm\sqrt{6},\dfrac{5}{2}\right)$

Now $|z+3i|=\sqrt{6+\dfrac{25}{4}}=\sqrt{\dfrac{49}{4}}=\dfrac{7}{2}=3.5$

54. Assume $z=a+ib\,(\text{where}\, b=164)$

By data $\dfrac{a+ib}{a+ib+n}=4i \Rightarrow a+ib=4i(a+ib+n)$

$\Rightarrow a=-46, 4(a+n)=b$

$\Rightarrow a=-4(164), 4(a=n)=164$

$\Rightarrow a=-656, n=697$

55. Assume $z=a+ib(b\neq 0)$, we have $(\text{Im}\, z)^5=b^5$

Now $\text{Im}\, z^5=5a^4b-10a^2b^3+b^5$

Observe $\dfrac{\text{Im}\, z^5}{(\text{Im}\, z)^5}=\dfrac{5a^4b-10a^2b^3+b^5}{b^5}$

$=5\left(\dfrac{a}{b}\right)^4=10\left(\dfrac{a}{b}\right)^2+1$

$=5x^2-10x+1$

$=5(x-1)^2-4 \rightarrow$Minimum value is - 4

56. Assume $\dfrac{1+z+z^2}{1-z+z^2}=\alpha$ (by data $\alpha\in R$)

Now $\alpha=\dfrac{(1-z+z^2)+2z}{(1-z+z^2)} \Rightarrow \alpha=1+\dfrac{2z}{1-z+z^2}$

$\Rightarrow \alpha-1=\dfrac{2z}{1-z+z^2} \Rightarrow \alpha-1=\dfrac{2}{\left(z+\dfrac{1}{z}\right)-1}$

$\Rightarrow \left(z+\dfrac{1}{z}\right)-1=\dfrac{2}{\alpha-1}$

$\Rightarrow z+\dfrac{1}{z}=\dfrac{2}{\alpha-1}+1 \rightarrow$Real (as $\alpha\in R$)

$\Rightarrow z+\dfrac{1}{z}=\left(z+\dfrac{1}{z}\right) \Rightarrow z+\dfrac{1}{z}=\bar{z}+\dfrac{1}{z}$

$\Rightarrow z-\bar{z}=\frac{1}{z}-\frac{1}{\bar{z}}$

$\Rightarrow z-\bar{z}=\frac{z-\bar{z}}{z\bar{z}} \Rightarrow z\bar{z}=\frac{z-\bar{z}}{z-\bar{z}}$

$\Rightarrow |z|^2=1 \ (\because z\neq\bar{z})$

$\Rightarrow |z|=1$

57. We have $z=18+26i$ and $|z|=10\sqrt{10}$, $\theta=\tan^{-1}\frac{13}{9}$

Now $z=r(\cos\theta+i\sin\theta)$

$z=10\sqrt{10}(\cos\theta+i\sin\theta)$ (Where $\tan\theta=\frac{13}{9}$)

$z^{\frac{1}{3}}=\sqrt{10}\left(\cos\frac{\theta}{3}+i\sin\frac{\theta}{3}\right)$ ..... (1)

Now $\tan\theta=\frac{13}{9} \Rightarrow \frac{3\tan\left(\frac{\theta}{3}\right)-\tan^3\left(\frac{\theta}{3}\right)}{1-3\tan^2\left(\frac{\theta}{3}\right)}=\frac{13}{9}$

$\Rightarrow \tan\frac{\theta}{3}=\frac{1}{3}$

$\Rightarrow \cos\frac{\theta}{3}=\frac{3}{\sqrt{10}}$, $\sin\frac{\theta}{3}=\frac{1}{\sqrt{10}}$ .... (2)

(2) in (1) $\Rightarrow z^{\frac{1}{3}}=\sqrt{10}\left(\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{10}}i\right)=\underbrace{3+i}_{x_0+iy_0}$

Thus $x_0=3, y_0=1 \Rightarrow \frac{x_0^3+y_0^3}{5}=\frac{28}{5}=5.6$

58. Given $zw=|z|^2 \Rightarrow zw=z\bar{z} \Rightarrow w=\bar{z}$

$\Rightarrow w=\bar{z}, \bar{w}=z$ .... (1)

We have $|z-\bar{z}|+|w+\bar{w}|=4 \Rightarrow |z-\bar{z}|+|z+\bar{z}|=4$

$[\because (1)]$

$\Rightarrow |ziy|+|zx|=4$

$(\because z=x+iy)$

$\Rightarrow |x|+|y|=2$ .... (2)

Area enclosed by (2) is 8 square units

59. We have $f(x)=ax^2+bx$

Now $f(1)=2, f(i^2)=0(f(-1)=0) \Rightarrow a+b=2, a-b=0$

$\Rightarrow a=1, b=1$

Also $f(x)=ax^2+bx \Rightarrow f(x)=x^2+x$

Now $f\left(z+\frac{1}{z}\right)=0 \Rightarrow \left(z+\frac{1}{z}\right)^2+\left(z+\frac{1}{z}\right)=0$

$\Rightarrow z^4+z^3+z^2+z^2+z+1=0$

$\Rightarrow (z^2+1)(z^2+z+1)=0$

$\Rightarrow (z+i)(z-i)(z^2+z+1)=0$

$\Rightarrow z = \pm i, z = w, w^2$ (Roots)

Sum of roots $= i + (-i) + w + w^2 = -1$

60. Given $x = i^2 + i\sqrt{2} \Rightarrow x+1 = i\sqrt{2} \Rightarrow (x+1)^2 = -2$

$\Rightarrow z^2 + 2x + 3 = 0$

Now $x^4 + 4x^3 + 6x^2 + 4x + 3 = (x^2 + 2x - 1)(x^2 + 2x + 3) + 6$

$= (x^2 + 2x - 1)(0) + 6 = 6$

61. Given $|z_1 - 1| = |z_2 - 1| = |z_3 - 1| = |z_4 - 1|$

$\Rightarrow z_1, z_2, z_3, z_4$ Are equidistant from 1

$\Rightarrow$ 1 is point of intersection of diagonals of square ABCD

$\Rightarrow$ mid points of AC, BD are same

(Diagonals bisect each over)

$\Rightarrow \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2} = 1$

$\Rightarrow z_1 + z_3 = 2, z_2 + z_4 = 2$

Adding $\Rightarrow (z_1 + z_3) + (z_2 + z_4) = 2 + 2$

$\Rightarrow z_1 + z_2 + z_3 + z_4 = 4$

62. Data $\Rightarrow z_1 + z_2 = -1, z_1 z_2 = \frac{\lambda}{2}$

Now 0, $A(z_1), B(z_2)$ forms equilateral triangle assumes

$z_1^2 + z_2^2 = 2z_1 z_2$

$\Rightarrow (z_1 + z_2)^2 - 2z_1 z_2 = z_1 z_2$

$\Rightarrow (z_1 + z_2)^2 = 3z_1 z_2 \Rightarrow (-1)^2 = 3\left(+\frac{\lambda}{2}\right)$

$\Rightarrow \frac{1}{\lambda} = \frac{3}{2} \Rightarrow \frac{1}{\lambda} = 1.5$

63. $|z| = 1 \Rightarrow x^2 + y^2 = 1$ ……… (1)

$\left|\frac{z}{z} + \frac{\bar{z}}{z}\right| = 1 \Rightarrow x^2 - y^2 = \pm\frac{1}{2}$ …….. (2)

Solving (1), (2), we get 8 possibilities for $[x, y]$

64. $f(x) = 5x^3 + Mx + N$ and $x^2 + x + 1 = (x - w)(x - w^2)$

By data $f(w) = 0, f(w^2) = 0$

$\Rightarrow 5 + Mw + N = 0, 5 + Mw^2 + N = 0$

Adding and subtracting $\Rightarrow M = 0, N = -5$

Thus $M + N = 5$

65. Observe $z^{100} = (z^3)^{14}(z^2) = z^2$ and $\frac{1}{z^{100}} = \frac{1}{z^2} = z^5$

$z^{300} = (2^7)^{42}(z^6) = z^6$ and $\frac{1}{z^{300}} = \frac{1}{z^6} = z$

$z^{500} = (z^7)^{71}(z^3) = z^3$ and $\frac{1}{z^{500}} = \frac{1}{z^3} = z^4$

Now $z^{100} + \frac{1}{z^{100}} + z^{300} + \frac{1}{z^{300}} + z^{500} + \frac{1}{z^{500}} = z^2 + z^5 + z^6 + z + z^3 + z^4$

$=\left(1+z+z^2+z^3+z^4+z^5+z^6\right)-1$

$=0-1=-1$

66. By data $z_1, z_2, z_3, ..., z_{50}$ are roots of $1+z+z^2+...+z^{50}=0$

$\Rightarrow 1+z+z^2+....+z^{50}=(z-z_1)(z-z_2)(z-z_3)....(z-z_{50})$

$\Rightarrow \log\left(1+z+z^2+....+z^{50}\right)=\log(z-z_1)+\log(z-z_2)+\log(z-z_3)+...+\log(z-z_{50})$

Diff. w.r. to z, we get

$$\frac{1+2z+3z^2+....+50z^{49}}{1+z+z^2+....+z^{50}}=\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}+...+\frac{1}{z-z_{50}}$$

Put $z=1$, we get

$$\frac{1+2+3+....+50}{51}=\frac{1}{1-z_1}+\frac{1}{1-z_2}+...+\frac{1}{1-z_{50}}$$

$$\Rightarrow \frac{51\times 50}{2\times 51}=-\left(\frac{1}{z_1-1}+\frac{1}{z_2-1}+...+\frac{1}{z_{50}-1}\right)$$

$$\Rightarrow \sum_{i=1}^{50}\frac{1}{z_i-1}=-25$$

67. Observe $\alpha=\dfrac{-1+i\sqrt{3}}{2}=w$

Given $(2+\alpha)^4+a+b\alpha \Rightarrow (2+w)^4=a+bw$

$\Rightarrow (a+bw)=(1+1+w)^4$

$=\left(1-w^2\right)^4 \left(\because 1+w+w^2=0 \Rightarrow 1+w=-w^2\right)$

$=1-4w^2+6w^4-4w^6+w^8$

$=1-4w^2+6w-4+w^2$

$=-3+6w-3w^2$

$=-3+6w-3(-1-w)=9w$

Thus $a+bw=9w \Rightarrow a=0, b=9 \Rightarrow a+b=9$

68. Let $A=\begin{bmatrix}1 & a & b\\ w & 2 & c\\ w^2 & w & 1\end{bmatrix} \Rightarrow |A|=(1-aw)(1-cw)$

Clearly $|A|$ is 0 when $a=w^2$ or $c=w^2$

$\therefore$ Possibilities to (a,b,c) are $\rightarrow 1\times 2\times 1=2$

69. We have $x^2-ix+1=0$

$\Rightarrow x^2+1=ix \Rightarrow x+\dfrac{1}{x}=i$ .. (1)

(1) $\Rightarrow \left(x+\dfrac{1}{x}\right)^2=i^2 \Rightarrow x^2+\dfrac{1}{x^2}=-3$ ….. (2)

(1) $\Rightarrow \left(x+\dfrac{1}{x}\right)^3=i^3 \Rightarrow x^3+\dfrac{1}{x^3}+3i=-i$

$\Rightarrow x^3+\dfrac{1}{x^3}=-4i$ …… (3)

Now from (2), (3) $\Rightarrow \left(x^2+\dfrac{1}{x^2}\right)\left(x^3+\dfrac{1}{x^3}\right)=(-3)(-4i)$

$\Rightarrow x^5 + \frac{1}{x^5} + x + \frac{1}{x} = 12i$

$\Rightarrow x^5 + \frac{1}{x^5} + i = 12i \Rightarrow x^5 + \frac{1}{x^5} = 11i$

$\Rightarrow \left(x^5 + \frac{1}{x^5} \right)^2 = (11i)^2$

$\Rightarrow x^{10} + \frac{1}{x^{10}} + 2 = -121 \Rightarrow x^{10} \Rightarrow x^{10} + \frac{1}{x^{10}} = -123$

70. Given $z^2 + |z|z + z\bar{z} = 0$ where $|z| = r$

$\Rightarrow z^2 + |z|z + |z|^2 = 0 \Rightarrow \frac{z^2}{|z|^2} + \frac{z}{|z|} + 1 = 0$

$\Rightarrow t^2 + t + 1 = 0$ (where $t = \frac{z}{|z|}$)

$\Rightarrow t = w, w^2 \Rightarrow \frac{z}{|z|} = w, w^2 \Rightarrow \frac{z}{r} = w, w^2$

$\Rightarrow z = rw, rw^2 \left(z_1, z_2 \right) \Rightarrow z_1 = rw, z_2 = rw^2$

$\Rightarrow |z_1 - z_2| = r\left|w - w^2\right|$

$\Rightarrow |z_1 - z_2| = r\left| \left(\frac{-1 + i\sqrt{3}}{2} \right) - \left(\frac{-1 - i\sqrt{3}}{2} \right) \right|$

$\Rightarrow |z_1 - z_2| = r\left|i\sqrt{3}\right| \Rightarrow |z_1 - z_2|^2 = 3r^2$

(we $|z_1 - z_2|^2 = \lambda r^2$)

$\Rightarrow \lambda = 3$